

Optimal Monetary Policy and Term Structure in a Continuous-Time DSGE Model*

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Abstract

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JEL Classification: C11, C62, E43, E52, G12

Keywords: Optimal monetary policy, Taylor rule, Term structure of interest rates, New Keynesian, Macroeconomic stability

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Abstract

We study optimal monetary policy, macro dynamics and their implications on the term structure of interest rates in a continuous-time New Keynesian model. With a quadratic cost function and regime-dependent monetary discount rates facing the central bank, the time-consistent optimal monetary policy is regime-dependent linear interest rate rules in inflation and output gaps. The optimal interest rate rules and the equilibrium dynamics of inflation and output gap form a regime-dependent term structure model. We take the model to the US data and find that the Fed had followed two distinct interest rate rules, one is not optimal while the other is near-optimal with a large monetary discount rate. The macro dynamics are more stable under the near-optimal policy rule than the non-optimal one.

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1 Introduction

Monetary policy and term structure of interest rates are probably two of the most closely related topics in economics and finance. On one hand, by setting the Fed fund target rate, the Federal Reserve not only anchors the short end of the yield curve but also influences future inflation and economic growth, which are key determinants of the term structure of interest rates. On the other hand, the forward-looking yield curve reflects market expectations of the future of the economy and could be an important indicator of the effectiveness of monetary policy. Despite such close connections, the macroeconomic literature on monetary policy and the finance literature on the term structure have been developed rather independently until recently.

The New Keynesian DSGE models have become the dominant approach used by both academics and central bankers for monetary policy analysis.¹ Under the sticky price equilibrium of these models, monetary policy is not neutral and has important impacts on the real activities of the economy and direct implications for the term structure. However, the existing macroeconomic literature on DSGE models has mainly focused on real activities and ignored term structure implications. Since the yield curve contains market expectations of future economic activities, we can better identify monetary policy by incorporating term structure data in the estimation of DSGE models. The term structure also provides an alternative perspective to examine potential shortcomings of DSGE models: If these models make counter-factual predictions on term structure dynamics, then one should be careful in using them for policy analysis.

The traditional dynamic term structure models (DTSMs) in the finance literature have primarily relied on *latent* variables to explain the evolution of the yield curve.² Whereas DTSMs are analytically tractable and empirically flexible, they do not clearly specify the economic nature of the latent variables. Therefore, term structure dynamics in DTSMs have not been explicitly linked to macroeconomic fundamentals.

The recent macro term structure literature tries to fill the gap by explicitly linking term structure dynamics to fundamental macroeconomic variables.³ Most models in this literature, how-

¹See, e.g., [Christiano et al. \(2005\)](#), [Galí and Gertler \(2007\)](#), [Smets and Wouters \(2007\)](#), and references therein.

²See [Dai and Singleton \(2003\)](#) for a comprehensive survey of the large literature on DTSMs.

³See, e.g., [Diebold et al. \(2005\)](#) and [Rudebusch \(2010\)](#) for surveys of this literature.

ever, are still partial equilibrium in nature, relying on exogenous assumptions on (i) the relation between the spot rate and macro variables and (ii) the dynamics of the macro variables. Assuming the spot rate follows an exogenous [Taylor \(1993\)](#) rule and inflation and output gaps follow Gaussian VAR process, [Ang and Piazzesi \(2003\)](#) introduce macro variables into traditional arbitrage-free affine term structure models.⁴ Although [Bekaert et al. \(2010\)](#) study term structure in a New Keynesian general equilibrium model, they also adopt a Taylor rule assumption. Without general equilibrium analysis of optimal monetary policy and dynamics of macro variables, most existing macro term structure models might not be able to fully capture the complex interactions among monetary policy, macro dynamics and the term structure of interest rates.

In this paper, we develop a continuous-time macro term structure model with optimal monetary policy under the New Keynesian DSGE framework.⁵ Continuous-time models have been widely used in the finance literature since the celebrated work of [Black and Scholes \(1973\)](#) and [Merton \(1971\)](#). Compared to existing optimal monetary policy literature and macro term structure models, our model has several important advantages and makes unique contributions to the literature.

We recast the simplest classic New Keynesian model in a continuous-time setting with complete market representation through state price. The resulting dynamics of inflation and output gaps are then controlled by monetary authority via nominal interest rate to minimize aggregate monetary losses weighted by a monetary discount rate over infinite horizon. The monetary loss at any instant is a quadratic function of inflation gap, output gap, and nominal interest rate, similar to the one used in [Woodford \(2003\)](#). The term with inflation and output gaps represents the welfare loss caused by sticky price and the nominal interest rate term represents direct control costs. A striking result of our model is that the optimal monetary policy is a linear interest rule in inflation and output gaps. By contrast, the optimal monetary policy in most discrete-time New Keynesian DSGE models in the macroeconomic literature are either too complicated or too simple

⁴[Ellingsen and Söderström \(2001\)](#) study optimal monetary policy and term structure of interest rates in a non-Keynesian model. However, long-term rates in their model are modeled directly as summation of expected future rates without imposing any arbitrage-free conditions.

⁵[Werning \(2012\)](#) and [Cochrane \(2014\)](#) use deterministic continuous-time New Keynesian model to study monetary policy and liquidity trap. There is no optimality consideration in [Cochrane \(2014\)](#) and the optimal policy in [Werning \(2012\)](#) is based commitment. In addition, these two studies do not link monetary policy with term structure of interest rates.

based on the commitment or discretion by monetary authority.⁶

The coefficients of the optimal linear interest rule is determined by the monetary cost function, which is a quadratic function in inflation and output gaps. The coefficient on inflation gap is positive and the coefficient on output gap is negative. All coefficients of the optimal rule converges to zero when the monetary discount rate goes to infinity. This suggests that a monetary authority may optimally adopt a zero or near zero policy rate when it extremely concerns about immediate macroeconomic outcomes. The optimal interest rate rule achieves macro stability if monetary authority weighs macroeconomic outcomes more balanced across time (low monetary discount rate). However, it violates macro stability conditions if the monetary authority is primarily concerned with immediate future outcomes (high monetary discount rate).

Usually, the optimal monetary policy cannot achieve perfect outcomes, i.e., zero inflation and output gaps, even asymptotically. The key reason for this result is the control costs that makes dramatic swings in nominal interest rate undesirable. When the control costs approach zero, the asymptotically perfect outcomes can be achieved. In this case, monetary authority can effectively control the output gap perfectly due to its great flexibility in adjusting the nominal interest rate.

Motivated by the dramatic difference in policy implications caused by different monetary discount rate, we extend our single regime model into a multiple regime model, in which a monetary authority may choose different monetary discount rate due to political pressure or concerns to put different weights to aggregate macroeconomic outcomes across time. The switching among regimes is governed by a continuous-time Markov chain and the optimal monetary policy is then regime-dependent. Conditional on regime, the optimal monetary policy is also a linear interest rate rule in inflation and output gaps similar to that of the single regime model.

With the optimal monetary policy is regime-dependent linear interest rate rules, the resulting dynamics of the inflation and output gaps also become regime-dependent. Together, they form a regime-dependent affine term structure of interest rate with inflation and output gaps as the state variables. Unlike existing macro term structure models with exogenously specified Taylor rule and dynamics of macro variables, ours is a general equilibrium macro term structure model with consistent optimal monetary policy and equilibrium evolution of the macro state variables.

⁶See [Erceg et al. \(2000\)](#) and [Clarida et al. \(1999\)](#) for commitment and discretion mechanism, and [Khan et al. \(2003\)](#) for the dynamic programming approach (numerical solutions).

The continuous-time approach yields a term structure model with closed-form solutions for bond prices. This tractability makes it very convenient to confront the model with the observed term structure data.

We confront the model to the US data, including US Treasury bond yields, GDP and GDP deflator, to study how the Fed conducts the monetary policy systematically in the past and what are the impacts of the monetary policy to the macro stability. We first estimate a model with a single interest rate rule. The single regime model does not perform well in explaining the observed Treasury yields. In addition, the estimated interest rate rule is not optimal. As suggested in [Li et al. \(2013\)](#), we also estimate a model with two switching monetary regimes.⁷ The two-regime model performs much better; it explains 81% of variations in Treasury yields. The empirical results are clearly in favor of the model with two monetary regimes. This is consistent with [Li et al. \(2013\)](#), who estimate a macro term structure model with regime-switching Taylor rules.

Our analysis provides new insights on US monetary policy and its impact on the real economy in the past half century. For example, we show that a single fixed linear interest rate rule does not capture bond yields well. Instead we find that the FED seems to have switched between two different interest rate policy rules. Our empirical studies reveal that there are two distinguished interest rate rules in the past: one is not optimal (e.g., the coefficient on output gap is positive); the other is near optimal if we assume that the Fed places greater weight on short-term macro variations. The near optimal policy is more stabilizing, especially for the output gap whereas the non-optimal policy is destabilizing for both inflation and output gap. The estimated policy regimes show that the near optimal policy started circa 1981 and ended circa 2001. Thus, the period when the Fed conducted the monetary policy (near) optimally is mostly overlapped with the “Great Moderation,” indicating that the better monetary policy had clearly contributed to the “Great Moderation.”

A key difference from the literature of monetary policy analyses, such as [Christiano et al. \(2005\)](#) and [Smets and Wouters \(2007\)](#) is that, given an exogenous Taylor rule, they focus on the impacts of monetary policy shocks on short-term dynamics of macro variables. Whereas we focus on the analyses of systematic policy rules and their impacts on long-term macro stability. Whereas

⁷Other studies on monetary policy regimes include [Sims and Zha \(2006\)](#) and [Bikbov and Chernov \(2014\)](#), the later also use term structure data.

the term structure data allows us to better understand and identify past monetary policies and macro dynamics that are consistent with a general equilibrium macro term structure model, which explicitly links term structure dynamics to macroeconomic fundamentals.

The rest of the paper is organized as follows. In Section 2, we introduce a continuous-time New Keynesian DSGE model with sticky prices. In Section 3, we derive and characterize the optimal monetary policy under the continuous-time DSGE model and the associated affine term structure model. In Section 4, we estimate our model using the US data and report the empirical results. Section 5 concludes, and the Appendix provides technical details and proofs.

2 Model

In this section, we introduce a continuous-time DSGE model with sticky prices. We first discuss optimization decisions facing households and firms. Then we solve for equilibrium under both flexible and sticky prices. In particular, we endogenously derive the equilibrium evolution of inflation and output gap, which are important inputs for further analysis of optimal monetary policy and the term structure of interest rates.

2.1 Households

A representative household has the following preference defined over a composite consumption good and labor

$$u(C, N) = E \left[\int_0^\infty e^{-\rho t} \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \frac{N_t^{1+\eta}}{1+\eta} \right) dt \right],$$

where ρ , γ , and η are constants and represent discount rate, degree of risk aversion, and aversion to labor, respectively, and C_t is the amount of consumption of the final consumption good, and N_t is the labor devoted by the household.

We assume that there are enough traded securities, e.g., bonds with different maturities, in the economy such that the markets are complete. Therefore, there exists a unique state price density to price all securities in equilibrium. Let ξ_t be the *nominal* state price density, defined by

$$\xi_t = \exp \left[- \int_0^t \left(r_s + \frac{1}{2} \|\lambda_s\|^2 \right) ds - \int_0^t \lambda_s \cdot dZ_s \right],$$

where r_s is the nominal interest rate, λ_s is the price of risk, and Z_t is a multi-dimensional standard Brownian motion. We use Z_t to represent shocks to productivity, output gap, aggregate price level, and the rate of inflation, which are defined later.

Let M_0 be the initial nominal market value of the firms owned by the representative household. Then the budget constraint for the household becomes

$$E \left[\int_0^\infty \xi_t (P_t C_t - W_t N_t) dt \right] \leq M_0,$$

where P_t is the price of the final consumption good, W_t is the wage for labor. Then the household's optimization problem can be solved by the *martingale approach* of [Cox and Huang \(1989\)](#):

$$\max_{\{C_t, N_t\}} E \left\{ \int_0^\infty e^{-\rho t} \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \frac{N_t^{1+\eta}}{1+\eta} \right) dt - \psi \left(\int_0^\infty \xi_t (P_t C_t - W_t N_t) dt - M_0 \right) \right\},$$

where ψ is the Lagrange multiplier for the budget constraint, or the shadow price of wealth. This optimization problem can be solved state-by-state and yields the following first-order conditions:

$$e^{-\rho t} C_t^{-\gamma} = \psi \xi_t P_t \tag{1}$$

for the composite consumption good, and

$$e^{-\rho t} \phi N_t^\eta = \psi \xi_t W_t \tag{2}$$

for the labor.

2.2 Final Good Producer

The final (composite) consumption good consists of a continuum of differentiated intermediate goods and is produced by a perfectly competitive firm according to the following technology

$$Y_t = \left(\int_0^1 C_{it}^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}, \tag{3}$$

where $\epsilon > 1$ is a constant.

Given the demand for the final good C_t , and the prices for the final good and each of the intermediate goods, P_{it} , the firm chooses C_{it} to maximize profit

$$\max_{C_{it}} \left\{ P_t Y_t - \int_0^1 P_{it} C_{it} di \right\}.$$

subject to the production function (3). Solving this problem yields the demand function for good i

$$C_{it} = \left(\frac{P_{it}}{P_t} \right)^{-\epsilon} Y_t. \quad (4)$$

This is the demand for inputs given the output Y_t . Since the final good producer faces perfect competition, its profit must be zero, that is

$$P_t Y_t = \int_0^1 P_{it} C_{it}.$$

Substituting the demand function in (4) yields the price below for the final consumption good

$$P_t = \left(\int_0^1 P_{it}^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}. \quad (5)$$

2.3 Firms and Monopolistic Competition

Following [Dixit and Stiglitz \(1977\)](#), we assume each differentiated consumption good is produced by a monopolistically competitive firm through a common production function

$$Y_{it} = A_t N_{it},$$

where A_t , a positive exogenously given stochastic process, represents productivity shocks, and N_{it} represents the labor devoted to the i th firm by the representative household. We assume that $a_t = \ln A_t$ satisfies

$$da_t = \mu_a dt + \sigma_a \cdot dZ_t,$$

where μ_a and σ_a are constants, and Z_t is the standard Brownian motion defined above.

Each firm's decision problem involves a two-stage optimization. First, fixing the demand, a

firm has to choose the amount of labor to minimize the costs:

$$\min_{N_{it}} \left\{ \frac{W_t}{P_t} N_{it} + \varphi_t (Y_{it} - A_t N_{it}) \right\},$$

where φ_t represents the real marginal cost of production. The first-order condition for this problem is

$$\varphi_t = \frac{W_t}{P_t} \frac{1}{A_t}. \quad (6)$$

This indicates that all firms have the same marginal cost of production. In the second stage, firms choose price to maximize the present value of profit streams, given the demand function (4).

In equilibrium, the markets for the intermediate goods and the final good must clear. That is, we must have

$$Y_{it} = C_{it} = A_t N_{it}, \text{ and } C_t = Y_t.$$

Then the clearing of labor markets implies

$$\int_0^1 N_{it} di = \frac{1}{A_t} \int_0^1 C_{it} di = \frac{Y_t}{A_t} \int_0^1 \left(\frac{P_{it}}{P_t} \right)^{-\epsilon} di = N_t.$$

Combining this with the first-order conditions (1) and (2), we have

$$\varphi_t = \phi \frac{Y_t^{\gamma+\eta}}{A_t^{1+\eta}} H_t^\eta, \quad (7)$$

where

$$H_t = \int_0^1 \left(\frac{P_{it}}{P_t} \right)^{-\epsilon} di \geq 1 \quad (8)$$

is the price dispersion across the intermediate goods.

2.4 Equilibrium under Flexible Prices

First, we solve for the equilibrium under the assumption that firms can adjust their prices without any costs. In this case, given the marginal cost (6), firms choose their prices to maximize real profit,

which can be done period-by-period as

$$\max_{P_{it}} \xi_t P_t \left(\frac{P_{it}}{P_t} C_{it} - \varphi_t C_{it} \right) = \max_{P_{it}} \xi_t P_t \left[\left(\frac{P_{it}}{P_t} \right)^{1-\epsilon} - \varphi_t \left(\frac{P_{it}}{P_t} \right)^{-\epsilon} \right] Y_t,$$

where the equality is obtained by the demand of good i given by (4). The first-order condition implies that

$$P_{it} = \frac{\epsilon}{\epsilon - 1} \varphi_t P_t.$$

Substituting this into the price index defined by (5), we have

$$\varphi_t = \bar{\varphi} \equiv \frac{\epsilon - 1}{\epsilon}.$$

This implies $P_{it} = P_t$ and $C_{it} = C_t$, and hence $H_t = 1$ by (8). Then equation (7) implies that the flexible-price equilibrium output is

$$\ln \bar{Y}_t = \frac{1 + \eta}{\gamma + \eta} \ln A_t - \frac{1}{\gamma + \eta} \ln \frac{\phi}{\bar{\varphi}} = \nu \ln A_t - \frac{1}{\gamma + \eta} \ln \frac{\phi}{\bar{\varphi}}, \quad (9)$$

where

$$\nu = \frac{1 + \eta}{\gamma + \eta}, \quad (10)$$

and \bar{Y}_t represents the flexible-price equilibrium output. In light of [Coibion and Gorodnichenko \(2011\)](#), we assume that the aggregate price has a constant trend or inflation target $\bar{\pi}$, i.e., $d \ln P_t = \bar{\pi} dt$, then, the market clearing condition $C_t = \bar{Y}_t$ leads to the price of risk

$$\bar{\lambda}_t = \gamma \nu \sigma_a \quad (11)$$

and the equilibrium nominal interest rate

$$\bar{r}_t = \rho + \bar{\pi} + \gamma \nu \mu_a - \frac{1}{2} \gamma^2 \|\nu \sigma_a\|^2. \quad (12)$$

Money or monetary policy is neutral in the flexible price equilibrium and has no impact on the real output.

2.5 Equilibrium under Sticky Prices

In reality, firms cannot adjust the prices of their products continuously. In this section, we study equilibrium under sticky prices based on the approach of [Calvo \(1983\)](#).

Specifically, we assume that over a period of Δt , there is a probability of $1 - e^{-\delta\Delta t}$ that a firm is able to reset the price of its product. Since all firms are identical except for their products, this probability also represents the fraction of firms that reset their prices over a period of Δt . As for the case of flexible prices, we assume the same inflation trend or target, $\bar{\pi}$, and firms are allowed to index their future price following the inflation trend. Given this real marginal cost and the demand function (4), firms that happen to reset price at t will choose price P_{it} to maximize

$$\begin{aligned} \max_{P_{it}} \frac{1}{\xi_t P_t} E_t \left[\int_t^\infty e^{-\delta(s-t)} \xi_s P_s \left(\frac{P_{it} e^{\bar{\pi}(s-t)}}{P_s} C_{is} - \varphi_s C_{is} \right) ds \right] \\ = \max_{P_{it}} \frac{1}{\xi_t P_t} E_t \left\{ \int_t^\infty e^{-\delta(s-t)} \xi_s P_s \left[\left(\frac{P_{it} e^{\bar{\pi}(s-t)}}{P_s} \right)^{1-\epsilon} - \varphi_s \left(\frac{P_{it} e^{\bar{\pi}(s-t)}}{P_s} \right)^{-\epsilon} \right] Y_s ds \right\}, \end{aligned}$$

where $e^{-\delta(s-t)}$ represents the probability that a firm does not reset its price over $s - t$.

As all firms are similar except for their products, all resetting firms at t choose the same price as

$$\frac{P_t^*}{P_t} = \frac{E_t \left[\int_t^\infty e^{-(\delta+\rho)(s-t)} Y_s^{1-\gamma} \frac{\varphi_s}{\bar{\varphi}} \left(\frac{P_s}{P_t e^{\bar{\pi}(s-t)}} \right)^\epsilon ds \right]}{E_t \left[\int_t^\infty e^{-(\delta+\rho)(s-t)} Y_s^{1-\gamma} \left(\frac{P_s}{P_t e^{\bar{\pi}(s-t)}} \right)^{\epsilon-1} ds \right]}, \quad (13)$$

where we use equation (1).

Since all resetting firms choose the same price, the price index defined as in equation (5) implies

$$E_t [P_{t+\Delta t}^{1-\epsilon}] \approx \delta \Delta t (P_{t+\Delta t}^*)^{1-\epsilon} + (1 - \delta \Delta t) (P_t e^{\bar{\pi} \Delta t})^{1-\epsilon}.$$

As $\Delta t \rightarrow 0$ and using the following approximations

$$\begin{aligned} \left(\frac{P_{t+\Delta t}}{P_t} \right)^{1-\epsilon} &\approx (1 - \epsilon) \ln \frac{P_{t+\Delta t}}{P_t} + 1, \\ \left(\frac{P_{t+\Delta t}^*}{P_t} \right)^{1-\epsilon} &\approx (1 - \epsilon) \frac{P_{t+\Delta t}^* - P_t}{P_t} + 1, \end{aligned}$$

$$e^{(1-\epsilon)\bar{\pi}\Delta t} \approx 1 + (1-\epsilon)\bar{\pi}\Delta t,$$

we have the expected local change of the logarithmic price index as

$$E_t[d \ln P_t] = \left(\delta \frac{P_t^* - P_t}{P_t} + \bar{\pi} \right) dt,$$

or in general we can rewrite this equation as

$$d \ln P_t = \left(\delta \frac{P_t^* - P_t}{P_t} + \bar{\pi} \right) dt + \sigma_p \cdot dZ_t = \pi_t dt + \sigma_p \cdot dZ_t, \quad (14)$$

where σ_p is a constant vector and π_t is the local expected rate of inflation, defined as

$$\pi_t \equiv \delta \frac{P_t^* - P_t}{P_t} + \bar{\pi}. \quad (15)$$

Let

$$\tilde{\rho} = \rho + \nu(\gamma - 1) \left(\mu_a - \frac{1}{2}[1 + \nu(\gamma - 1)]\|\sigma_a\|^2 \right), \quad (16)$$

which has to be positive for the utility to be finite in equilibrium. We assume that the shocks to the real and nominal sides are independent. As shown in Appendix A.1, applying linear approximations to equation (13) around the flexible equilibrium yields

$$\frac{P_t^* - P_t}{P_t} = (\delta + \tilde{\rho}) E_t \left[\int_t^\infty e^{-(\delta + \tilde{\rho})(s-t)} \left(\ln \frac{\varphi_s}{\bar{\varphi}} + \ln \frac{P_s}{P_t} - \bar{\pi}(s-t) \right) ds \right]. \quad (17)$$

Then, by equation (15), we have

$$\begin{aligned} \frac{1}{\delta} E_t[d\pi_t] &= \left(\frac{\tilde{\rho} + \delta}{\delta} (\pi_t - \bar{\pi}) - (\delta + \tilde{\rho}) \ln \frac{\varphi_t}{\bar{\varphi}} \right) dt \\ &\quad - (\delta + \tilde{\rho}) E_t \left[\int_t^\infty e^{-(\delta + \tilde{\rho})(s-t)} [(\pi_t - \bar{\pi}) dt + \sigma_p \cdot dZ_t] ds \right]. \end{aligned}$$

Using the following identity, directly implied by equation (7), and $\ln H_t \approx 0$,

$$\ln \frac{\varphi_t}{\bar{\varphi}} \approx (\gamma + \eta) \ln \frac{Y_t}{\bar{Y}_t},$$

we have the *New Keynesian Phillips curve* in continuous-time⁸

$$E_t[d\pi_t] = \left(\tilde{\rho}(\pi_t - \bar{\pi}) - \delta(\delta + \tilde{\rho}) \ln \frac{\varphi_t}{\bar{\varphi}} \right) dt = \left(\tilde{\rho}(\pi_t - \bar{\pi}) - \kappa_y \ln \frac{Y_t}{\bar{Y}_t} \right) dt,$$

where $\kappa_y = \delta(\delta + \tilde{\rho})(\gamma + \eta) > 0$ because $\tilde{\rho} > 0$. In general, this equation can be rewritten as

$$d\pi_t = [\tilde{\rho}(\pi_t - \bar{\pi}) - \kappa_y(y_t - \bar{y}_t)] dt + \sigma_\pi \cdot dZ_t, \quad (18)$$

where $y_t = \ln Y_t$, $\bar{y}_t = \ln \bar{Y}_t$, and σ_π is a constant vector.

Substituting $C_t = Y_t$ into equation (1) and rewrite it as

$$dy_t = \frac{1}{\gamma} \left(r_t - \rho - \pi_t + \frac{1}{2} \|\lambda_t\|^2 \right) dt + \frac{1}{\gamma} (\lambda_t - \sigma_p) \cdot dZ_t \quad (19)$$

and the output under flexible price as

$$d\bar{y}_t = \nu da_t = \nu [\mu_a dt + \sigma_a \cdot dZ_t].$$

Let $x_t = y_t - \bar{y}_t$ be the output gap, then

$$dx_t = \frac{1}{\gamma} [r_t - \tilde{r} - (\pi_t - \bar{\pi})] dt + \sigma_x \cdot dZ_t, \quad (20)$$

where

$$\tilde{r} = \bar{\pi} + \rho + \gamma \nu \mu_a - \frac{1}{2} \|\lambda_t\|^2, \quad (21)$$

and

$$\sigma_x = \frac{1}{\gamma} (\lambda_t - \sigma_p - \gamma \nu \sigma_a) \quad (22)$$

is the volatility vector of the output gap. The independence between the real and nominal shocks

⁸It is straightforward to obtain a discrete-time version as

$$E_t[\pi_{t+\Delta t}] - \pi_t = \tilde{\rho} \Delta t [\pi_t - \bar{\pi}] - \kappa_y \Delta t x_t.$$

Thus, we have

$$\pi_t - \bar{\pi} = \frac{1}{1 + \tilde{\rho} \Delta t} E_t[\pi_{t+\Delta t} - \bar{\pi}] + \frac{\kappa_y \Delta t}{1 + \tilde{\rho} \Delta t} x_t.$$

This is known as the New Keynesian Phillips curve, e.g., see Galí (2008).

is satisfied if $\sigma_a \cdot \sigma_\pi = \sigma_a \cdot \sigma_p = \sigma_a \cdot \sigma_x = 0$. These assumptions imply

$$\|\lambda_t\|^2 = \|\sigma_p + \gamma\sigma_x\|^2 + \gamma^2\nu^2\|\sigma_a\|^2,$$

thus, by equation (21),

$$\tilde{r} = \bar{\pi} + \rho + \gamma \left(\nu\mu_a - \frac{1}{2}\gamma\nu^2\|\sigma_a\|^2 \right) - \frac{1}{2}\|\sigma_p + \gamma\sigma_x\|^2. \quad (23)$$

With a specification of the nominal interest rate r_t , the dynamics of inflation (18) and output gap (20) form an equilibrium dynamic system as

$$d \begin{pmatrix} \pi_t \\ x_t \end{pmatrix} = \begin{pmatrix} \tilde{\rho}(\pi_t - \bar{\pi}) - \kappa_y x_t \\ \frac{1}{\gamma} [r_t - \tilde{r} - (\pi_t - \bar{\pi})] \end{pmatrix} dt + \begin{bmatrix} \sigma_\pi^\top \\ \sigma_x^\top \end{bmatrix} dZ_t. \quad (24)$$

That the equilibrium depends on the nominal interest rate is the key feature of the New Keynesian DSGE model. Thus, monetary policy plays an important role in shaping the equilibrium.

3 Optimal Monetary Policy and Term Structure

In this section, we study optimal monetary policy under our continuous-time DSGE model. Following Woodford (2003), we assume that the monetary authority seeks to choose the short-term nominal interest rate r_t to minimize a quadratic loss function of the following form

$$E_t \left[\frac{1}{2} \int_t^\infty e^{-\rho_m(s-t)} \left((\pi_s - \bar{\pi})^2 + \alpha_x x_s^2 + \alpha_r r_s^2 \right) ds \right], \forall t \geq 0,$$

subject to the dynamics of inflation and output gap under the sticky price equilibrium in (24). In the loss function, $\rho_m > 0$ is the monetary discount rate⁹ for the monetary authority, and $\alpha_x > 0$ and $\alpha_r > 0$ are the weights put on the variations of output gap and nominal interest rate, respectively.

The variations in inflation and output gap are related to the welfare loss and the variation in the

⁹The central bank's discount rate is usually set as the same as the one of representative consumer. However, we allow it to be different to model central bank's relative independent objective, e.g., a large ρ_m for accommodating the political pressure to solve short-term issues.

nominal interest rate is related to the control costs. The dynamic system of the two state variables, inflation π_t and output gap x_t , follows a Markovian structure if the nominal interest rate is a function of current state variables. Thus, the optimization problem can be solved by dynamic programming.

3.1 Optimal Monetary Policy

We first analyze the case in which the discount rate for the monetary authority is constant over time, then we show that many properties of the optimal monetary policy under single regime can be extended into the optimal monetary policies with regime-dependent time discount rates.

3.1.1 Single Monetary Policy Regime

Let $V(\pi_t, x_t)$ be the cost function, that is¹⁰

$$V(\pi_t, x_t) = \min_{\{r_s\}} E_t \left[\frac{1}{2} \int_t^\infty e^{-\rho_m(s-t)} \left((\pi_s - \bar{\pi})^2 + \alpha_x x_s^2 + \alpha_r r_s^2 \right) ds \mid \text{subject to (24)} \right]. \quad (25)$$

This is a standard dynamic programming problem. The Hamilton-Jacob-Bellman (HJB) equation associated with this optimization problem is:

$$\begin{aligned} -\rho_m V + \min_r \left\{ \left(\tilde{\rho}(\pi - \bar{\pi}) - \kappa_y x \right) \frac{\partial V}{\partial \pi} + \frac{1}{2} \|\sigma_\pi\|^2 \frac{\partial^2 V}{\partial \pi^2} \right. \\ \left. + \frac{1}{\gamma} \left(r - \tilde{r} - (\pi - \bar{\pi}) \right) \frac{\partial V}{\partial x} + \frac{1}{2} \|\sigma_x\|^2 \frac{\partial^2 V}{\partial x^2} + \sigma_x \cdot \sigma_\pi \frac{\partial^2 V}{\partial \pi \partial x} \right. \\ \left. + \frac{1}{2} \left((\pi - \bar{\pi})^2 + \alpha_x x^2 + \alpha_r r^2 \right) \right\} = 0. \end{aligned} \quad (26)$$

Given the quadratic form, the first-order condition is necessary and sufficient,

$$r^* = -\frac{1}{\alpha_r \gamma} \frac{\partial V}{\partial x}.$$

This interest rule yields the optimal monetary policy after we solve the cost function $V(\pi, x)$.

¹⁰The optimization problem loses its tractability with a non-negative nominal interest rate constraint. The optimization problem can only be solved numerically and so does the optimal policy.

We conjecture a solution to the cost function as

$$V(\pi, x) = \frac{1}{2} \left(A_\pi (\pi - \bar{\pi})^2 + A_x x^2 \right) + B_{\pi x} x (\pi - \bar{\pi}) + C_x x + C_\pi (\pi - \bar{\pi}) + D, \quad (27)$$

where $A_\pi > 0$, $A_x > 0$, $B_{\pi x}$, C_π , C_x , and D are constant. Therefore, the optimal interest rate rule is

$$r^* = \beta_0^* + \beta_\pi^* (\pi - \bar{\pi}) + \beta_x^* x, \quad (28)$$

where

$$\beta_0^* = -\frac{1}{\gamma \alpha_r} C_x, \quad \beta_\pi^* = -\frac{1}{\gamma \alpha_r} B_{\pi x}, \quad \beta_x^* = -\frac{1}{\gamma \alpha_r} A_x. \quad (29)$$

To solve for the coefficients of the optimal interest rate rule, we substitute r^* and the derivatives of V into the HJB equation. Given that the HJB equation must hold for different values of π and x , all its coefficients on $(\pi - \bar{\pi})^2$, x^2 , $(\pi - \bar{\pi})x$, $\pi - \bar{\pi}$, x , and the constant have to be exactly zero. This leads to six nonlinear equations (see Appendix A.2.1) that the coefficients of the cost function must satisfy. Solving this system of equations yields the cost function, and hence the optimal monetary policy rule as expressed in (28). This also verifies the conjectured cost function.

The optimal interest rate rule is linear in the state variables—inflation and output gaps. This is the direct consequence of the linear dynamics of the state variables and the quadratic form of the loss function. This is in dramatic contrast to the majority of existing studies on optimal monetary policy in the literature of monetary economics. Most of these studies are based on discrete-time DSGE models and tend to generate much more complicated optimal policy rules. In fact, [Woodford \(2001\)](#) states that¹¹ “Of course, it will surprise no one that such a simple rule (*the original Taylor rule*, added by the authors) is unlikely to correspond to fully optimal policy in the context of a particular economic model.” Instead of using the Lagrangian multiplier approach as, for example, in [Woodford \(2003\)](#), we use the continuous-time dynamic programming approach based on the HJB equation. Our results show that linear interest rate rule can be optimal.

Certain properties of the optimal policy can be studied without explicitly solving the policy itself.

¹¹This quote is in the working paper version.

Proposition 1 *The optimal monetary policy is a linear interest rate rule in inflation and output gap. The coefficient of the optimal interest rate rule on the output gap, β_x^* , is negative. The coefficient of the optimal interest rule of inflation, β_π^* , is greater than one if $\tilde{\rho} \leq \rho_m \leq 2\tilde{\rho}$.*

Since A_x must be positive for optimality, the coefficient of output gap in the optimal interest rule (28) must be negative. This is a robust feature of the optimal rule and does not depend on any specific set of parameters as long as the optimal problem has a solution. The negative coefficient of the output gap is clearly different from that of the famous Taylor rule, in which the coefficient of output gap is positive. This might appear counter-intuitive, because the central bank should raise interest rate when the output gap is positive to slow down the economy, and *vice versa*. This result shows intuition may not hold in an equilibrium model and highlights the importance and advantage of general equilibrium analysis.

As shown in the Appendix, the condition $\tilde{\rho} \leq \rho_m \leq 2\tilde{\rho}$ implies

$$\beta_\pi^* = -\frac{1}{\gamma\alpha_r} B_{\pi x} = 1 + \sqrt{1 + \frac{1 + (2\tilde{\rho} - \rho_m)A_\pi}{\alpha_r}} > 1.$$

Thus, the optimal interest rate rule satisfies the *Taylor principle*, where the coefficient of inflation is greater than one if the discount rate used by the central bank is mild. However, as shown by numerical examples below, the condition $\tilde{\rho} \leq \rho_m \leq 2\tilde{\rho}$ is not necessary for the *Taylor principle* to hold. In fact, the *Taylor principle* holds for the optimal interest rate rule for a much wide range of the central bank's discount rate. The previous equation also shows that $B_{\pi x}$ in the cost function is negative, which means that it is very costly when the output gap x and inflation π have opposite signs, e.g., high inflation and negative output gap.

Since the optimal monetary policy is a linear interest rate rule, we examine the conditions for a general linear interest rate rule that make macro dynamics stable. Given an arbitrary interest rate rule $r_t = \beta_0 + \beta_\pi(\pi_t - \bar{\pi}) + \beta_x x_t$, the joint dynamics of inflation and output gap (24) become

$$d \begin{pmatrix} \pi_t \\ x_t \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ \frac{1}{\gamma}(\beta_0 - \tilde{r}) \end{pmatrix} + \begin{bmatrix} \tilde{\rho} & -\kappa_y \\ \frac{1}{\gamma}(\beta_\pi - 1) & \frac{1}{\gamma}\beta_x \end{bmatrix} \begin{pmatrix} \pi_t - \bar{\pi} \\ x_t \end{pmatrix} \right\} dt + \begin{bmatrix} \sigma_\pi^\top \\ \sigma_x^\top \end{bmatrix} dZ_t$$

$$\equiv \left\{ \theta_P + \kappa \begin{pmatrix} \pi_t - \bar{\pi} \\ x_t \end{pmatrix} \right\} dt + \sigma dZ_t. \quad (30)$$

It is straightforward to verify that this stochastic differential equation (SDE) has a solution as

$$\begin{pmatrix} \pi_t - \bar{\pi} \\ x_t \end{pmatrix} = e^{t\kappa} \left\{ \begin{pmatrix} \pi_0 - \bar{\pi} \\ x_0 \end{pmatrix} + \int_0^t e^{-s\kappa} \theta_P ds + \int_0^t e^{-s\kappa} \sigma dZ_s \right\}.$$

From this we can compute the expected values and the average covariance of the macro variables (e.g., see [Karatzas and Shreve \(1991\)](#)) as

$$E \left[\begin{pmatrix} \pi_t - \bar{\pi} \\ x_t \end{pmatrix} \right] = e^{t\kappa} \left\{ \begin{pmatrix} \pi_0 - \bar{\pi} \\ x_0 \end{pmatrix} + \int_0^t e^{-s\kappa} \theta_P ds \right\}, \quad (31)$$

and

$$\text{Cov} \left[\begin{pmatrix} \pi_t - \bar{\pi} \\ x_t \end{pmatrix} \right] = \frac{1}{t} \int_0^t e^{(t-s)\kappa} \sigma \sigma^\top e^{(t-s)\kappa^\top} ds. \quad (32)$$

The stability of the macro system (inflation and output gap) can be examined from two different perspectives. One is the ability of the system in absorbing shocks to inflation and output gap, e.g., how expected values of π_t and x_t evolve given their current values. The other is how well the system absorbs future shocks, that is, how the variance-covariance function depends on time. As indicated by equations (31) and (32), the stability of the macro system from both perspectives hinges on the properties of the κ -matrix

$$\kappa = \begin{bmatrix} \tilde{\rho} & -\kappa_y \\ \frac{1}{\gamma}(\beta_\pi - 1) & \frac{1}{\gamma}\beta_x \end{bmatrix}. \quad (33)$$

Specifically, if the real parts of the two eigenvalues of κ are negative, then inflation and output gap would have a stationary distribution.

Proposition 2 *Given an arbitrary monetary policy $r_t = \beta_0 + \beta_\pi(\pi_t - \bar{\pi}) + \beta_x x_t$, the necessary and*

sufficient conditions for macro stability are

$$\beta_\pi > 1 - \frac{\tilde{\rho}}{\kappa_y} \beta_x, \quad \beta_x < -\gamma \tilde{\rho}, \quad (34)$$

where $\tilde{\rho}$ is defined in (16). If both conditions in (34) hold, $\lim_{t \rightarrow \infty} E[(\pi_t - \bar{\pi}, x_t)^\top] = -\kappa^{-1} \theta_P$.

The dynamics of inflation tends to be destabilizing given that inflation tends to increase (decrease) when it is high (low) because $\tilde{\rho}$ is positive. The system can only drive inflation down via output gap. Thus, the dynamics of output gap has to be mean-reverting enough for stability. This can only be achieved by a monetary policy rule that has a relatively large negative coefficient for output gap. When the *Taylor principle* holds, higher inflation also tends to push output gap higher, which in turn pushes inflation lower. However, as shown by Proposition 2, *Taylor principle* alone is not sufficient to guarantee macro stability.

The following proposition gives a condition under which the zero interest rate is optimal.

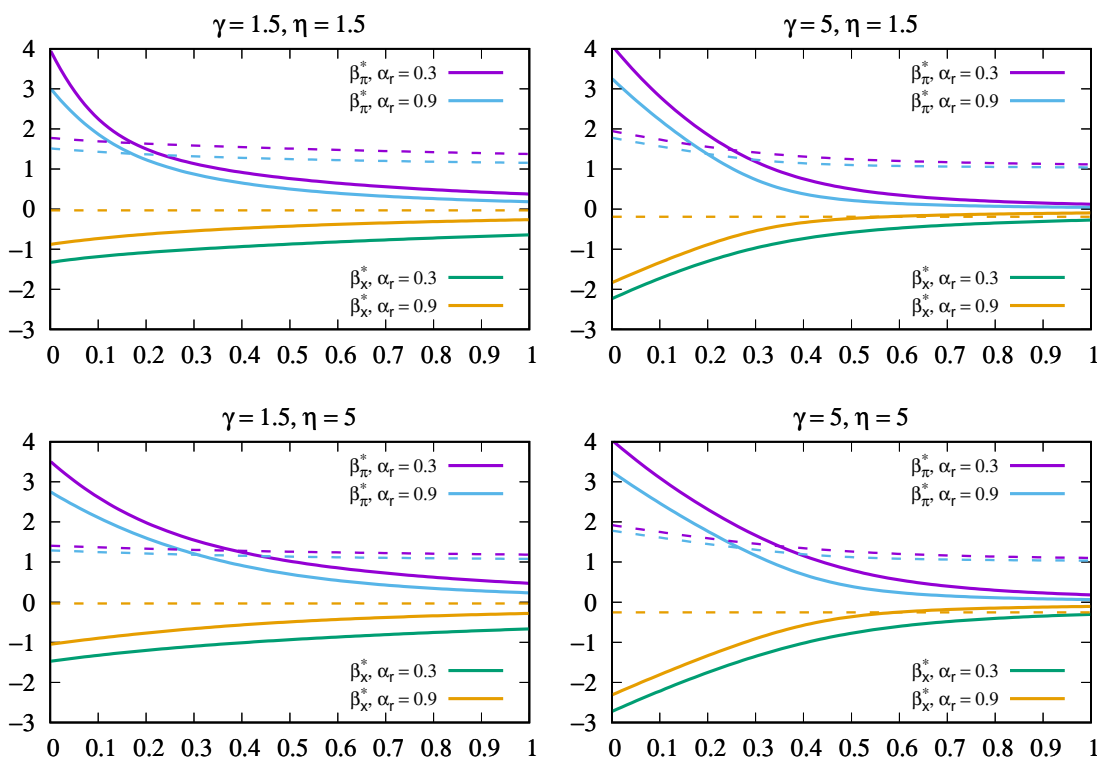
Proposition 3 *If the monetary discount rate, ρ_m , goes to infinity, then the coefficients of the optimal interest rule given by (28) all converge to zero.*

When a monetary authority worries so much about current variations of inflation and output gap, they may “optimally” adopt a zero or near-zero interest rate policy even though the zero rate policy is destabilizing by Proposition 2. This offers an alternative explanation for the zero or near-zero interest rate policy taken by the US Fed since 2008.

Now we use numerical solutions to further explore the properties of the optimal interest rate rule. As stated earlier, solving the cost function and hence the optimal monetary policy rule involves solving a system of 6 nonlinear equations as described in Appendix A.2.1.

Figure 1 plots the coefficients of inflation and output gap, as well as their macro stable conditions, in the optimal interest rate rule over a range of monetary discount rate under different model parameters. A key feature is that the magnitudes of both coefficients on output gap and inflation always decrease with monetary discount rate across different preference and monetary control cost. This implies that, if the monetary authority worries about near term stability of the economy, that is, a high monetary discount rate (ρ_m), then the resulting optimal monetary policy

Figure 1: **Optimal Monetary Policy: Coefficients of Inflation and Output Gap vs Monetary Discount Rate (ρ_m)**

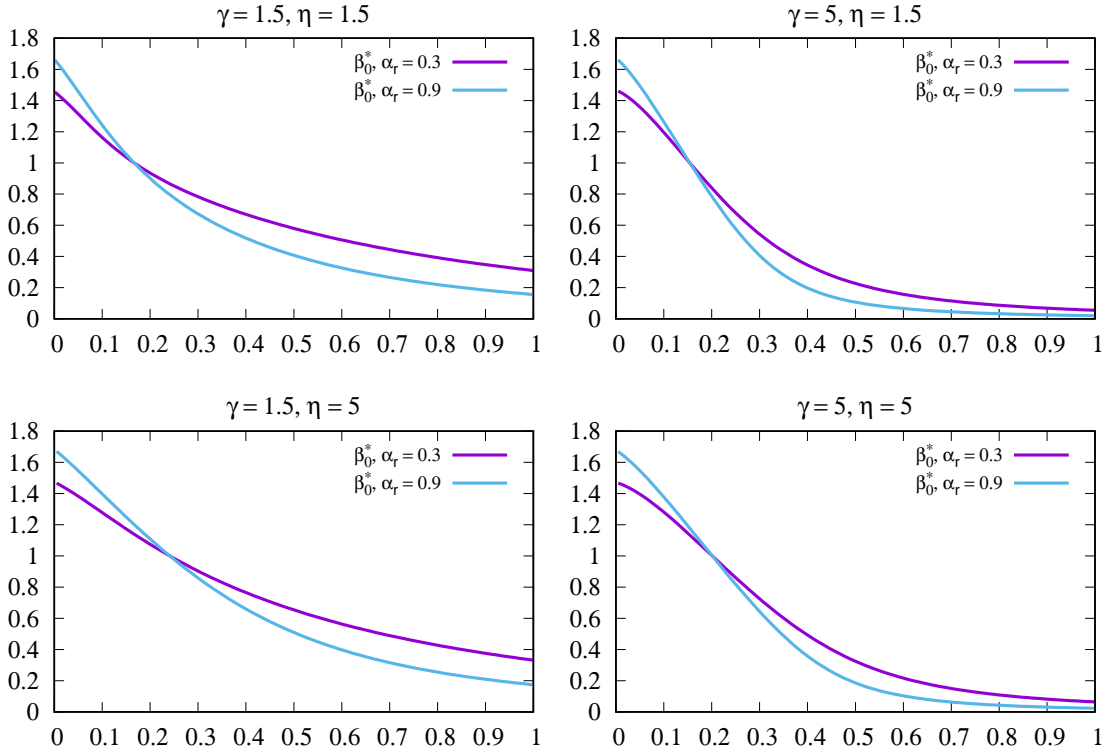


This figure provides the coefficients of inflation and output gap for the optimal interest rule for a range of monetary discount rate. The parameters are set as follows: $\rho = 1.5\%$, $\delta = 10$, $\mu_a = 1.5\%$, $\sigma_a = 0$, and $\alpha_x = 0.4$. Parameters that do not affect monetary policy are omitted. The color-matched dash lines represent the lower (upper) bounds of β_π^* (β_x^*) for macro stability. Notice that the two upper bounds of β_x^* are the same.

rule has smaller coefficients on output gap and inflation.

Another feature revealed by these numerical examples is that macro stability holds for quite large monetary policy discount rate and the condition on the inflation coefficient is much tighter than the one on the output gap coefficient. However, macro stability fails for high monetary discount rate, which implies the monetary policy authority is more greatly concerned with near-term outcomes than long-term ones. Thus, if the monetary authority switches its priority between long-term and short-term stability, we would expect that the resulting optimal policy rule switches as indicated by the changes of the coefficients. This prediction is consistent with empirical evidence on regime switching in the Taylor rule as documented in [Li et al. \(2013\)](#). The problems with multiple policy regimes are formally addressed in the next subsection.

Figure 2: Optimal Monetary Policy: Constant vs Monetary Discount Rate (ρ_m)

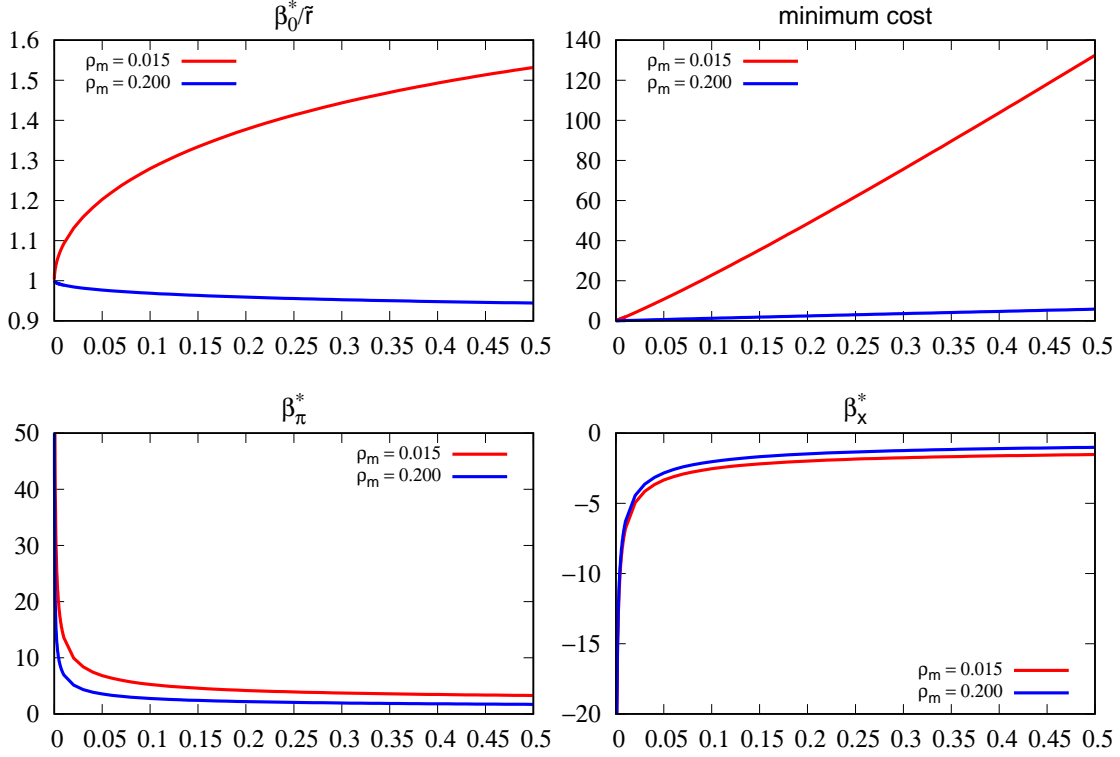


This figure provides the constant term normalized by \tilde{r} , β_0^*/\tilde{r} , for the optimal interest rule for a range of monetary discount rate, where \tilde{r} is as defined in equation (23). The parameters are set as follows: $\rho = 1.5\%$, $\delta = 10\%$, $\mu_a = 1.5\%$, $\sigma_a = 0$, and $\alpha_x = 0.4$. Parameters that do not affect monetary policy are omitted.

Moreover, both the constant term, shown in Figure 2, and the coefficients of inflation and output gap in the optimal interest rate rule converge to zero when ρ_m goes to infinity as predicted by Proposition 3. If the monetary authority is heavily concerned with the immediate outcome, it may be optimal to choose a zero interest rate. For example, the US Fed and many other central banks have taken zero or near zero interest rate after the 2008 financial crisis. Our model may offer an alternative explanation for the observed zero rate to the traditional explanation: nominal interest rate hits its lower zero bound.

A somewhat surprising result from these numerical examples is that β_0^*/\tilde{r} is far from 1 when the optimal monetary policy achieves macro stability (i.e., for relatively low monetary discount rate). This means a stabilizing optimal monetary policy may well not achieve the ideal outcomes — both zero inflation and output gaps — asymptotically because the asymptotic means are given by $(\kappa^*)^{-1}[0, (\beta_0^* - \tilde{r})/\gamma]^\top$, which does not equal zero when $\beta_0^* \neq \tilde{r}$, by Proposition 2. Figure 2 seems

Figure 3: Effects of Control Cost (α_r)



This figure depicts the effects of control cost on the optimal interest rule and cost function given by (27). The parameters are set as follows: $\gamma = \eta = 3$, $\rho = 1.5\%$, $\delta = 10\%$, $\mu_a = 1.5\%$, $\sigma_a = 0$, and $\alpha_x = 0.4$. Parameters that do not affect monetary policy are omitted. The minimum cost represents the minimum of the monetary cost function minus the constant term and is normalized by \bar{r}^2 . β_π and β_x are truncated for clear comparison.

to suggest that lowering the control cost (α_r) pulls β^*/\bar{r} close to 1 as β^*/\bar{r} is closer to 1 for $\alpha_r = 0.3$ than that for $\alpha_r = 0.9$ in general.

A related question is what inflation and output gap are when the monetary cost function reaches its minimum? That is $(\pi^*, x^*) = \arg \min_{\{\pi, x\}} V(\pi, x)$. Since the monetary cost function V is quadratic and globally convex, and hence has a global minimum, the minimization is straightforward. The solution is given in Appendix A.2.2. Our numerical experiments indicate that $(\pi^* - \bar{\pi}, x^*)$ and $(\kappa^*)^{-1}[0, (\beta_0^* - \bar{r})/\gamma]^\top$ are different and both of them are different from zeros. Figure 3 plots coefficients of the optimal monetary policy and the minimum of the monetary cost function without the constant part, which is positive, for a range of the control cost α_r . As Figure 3 shown, β^*/\bar{r} does converge to one and the minimum of the variable monetary cost function decreases to zero as the control cost goes to zero. This is quite intuitive: monetary policy instrument

— the nominal interest rate — reaches its maximum power and flexibility when the associated control cost is zero. This is indeed the case as shown by the magnitudes of the coefficients of inflation and output gap in the policy rule increase dramatically as the control cost goes to zero. The following proposition shows those numerical observations formally.

Proposition 4 *As α_r goes to zero, the monetary cost function as defined by (25) converges to*

$$\lim_{\alpha_r \rightarrow 0} V(\pi, x) = \frac{1}{2} \bar{A}_\pi (\pi - \bar{\pi})^2 + \frac{\bar{A}_\pi}{2\rho_m} \|\sigma_\pi\|^2,$$

and the associated optimal monetary policy is given by

$$r^* = \begin{cases} \tilde{r}, & \text{if } x = \frac{\kappa_y \bar{A}_\pi}{\alpha_x} (\pi - \bar{\pi}) \\ \infty, & \text{if } x > \frac{\kappa_y \bar{A}_\pi}{\alpha_x} (\pi - \bar{\pi}) \\ -\infty, & \text{if } x < \frac{\kappa_y \bar{A}_\pi}{\alpha_x} (\pi - \bar{\pi}), \end{cases} \quad (35)$$

where $\bar{A}_\pi = \lim_{\alpha_r \rightarrow 0} A_\pi$ is explicitly defined by (64) in Appendix A.2.3.

As shown in Appendix A.2.3, as the control cost α_r goes to zero, β_0^* converges to \tilde{r} , β_π^* goes to infinity, and β_x^* goes to negative infinity. Even though the later coefficients become unbounded, their ratio $\beta_\pi^*/\beta_x^* = A_\pi/A_x$ does converge to a constant, $-\kappa_y \bar{A}_\pi/\alpha_x$, which is the key factor to determine the optimal control r^* given by (35). If the control cost is zero, the optimal control problem becomes a bang-bang control, in which the optimal control brings the output gap back on track instantly whenever it wanders off. Thus, the optimal control — nominal interest rate — should remain a constant (\tilde{r}) with occasionally instant large spikes.

Proposition 4 is also relevant to the existing literature in optimal monetary policy, in which it is a common practice to assume that a monetary authority is able to control output gap directly. In this case, the optimal control problem (25) becomes

$$V_0(\pi_t) = \min_{\{x_t\}} E_t \left[\frac{1}{2} \int_t^\infty e^{-\rho_m(s-t)} \left((\pi_s - \bar{\pi})^2 + \alpha_x x_s^2 \right) ds \right] \quad (36)$$

subject to the dynamics of inflation

$$d\pi_t = \left(\tilde{\rho}(\pi_t - \bar{\pi}) - \kappa_y x_t \right) dt + \sigma_\pi^\top dZ_t.$$

Following a similar procedure as that for optimal control problem (25), Appendix A.2.4 shows the following results.

Proposition 5 *The monetary cost function V_0 is exactly the same as that defined in Proposition 4. That is*

$$V_0(\pi) = \lim_{\alpha_r \rightarrow 0} V(\pi, x) = \frac{1}{2} \bar{A}_\pi (\pi - \bar{\pi})^2 + \frac{\bar{A}_\pi}{2\rho_m} \|\sigma_\pi\|^2.$$

The optimal control policy is given by

$$x = \frac{\kappa_y \bar{A}_\pi}{\alpha_x} (\pi - \bar{\pi}),$$

where \bar{A}_π is given by (64) in Appendix A.2.2.

This shows the original optimization problem with nominal interest rate as the control instrument is equivalent to an alternative optimal control problem with the output gap as the control instrument when the control cost associated with nominal interest rate is zero. On the other hand, our results offer insights and mechanism on how a monetary authority can achieve to directly control output gap. It is also interesting to notice that when the direct control cost associated with output gap goes to zero, that is $\alpha_x \rightarrow 0$, \bar{A}_π converges to zero. Thus, the outcomes of the optimal policy for this case are the best: both inflation and output gaps are zero. This also illustrates on why the ideal outcomes cannot be achieved in general when α_x and α_r are not zero.

3.1.2 Multiple Monetary Policy Regimes

Our theoretical analysis for the case of single policy regime shows that different monetary discount rate ρ_m can lead to different monetary policy rules with different coefficients of inflation and output gap. It is conceivable that a monetary authority may be more concerned about current and near-term state of the economy (high ρ_m) or weight concerns evenly across all terms (low ρ_m), and these concerns may change over time depending on the state of the economy and political

pressure. Previous studies, e.g., [Li et al. \(2013\)](#), have shown that US monetary policy during 1952-2007 had followed distinct regimes. In this section, we extend our single-regime model to multiple monetary policy regimes caused by regime-switching in the monetary discount rate.

Suppose ρ_m switches among K values ($\rho_{m1} < \dots < \rho_{mK}$), the switching among the K regimes is governed by a continuous-time Markov chain with a constant transition rate matrix Q . Let

$$A = Q - \begin{bmatrix} \rho_{m1} & \cdots & 0 \\ \cdots & \rho_{mk} & \cdots \\ 0 & \cdots & \rho_{mK} \end{bmatrix}. \quad (37)$$

Notice that A is a constant matrix and has K distinct negative eigenvalues, Λ_{kk} for $k = 1, \dots, K$. Since the monetary discount rate is time-varying, the central bank's objective is to set nominal interest rate under each regime to minimize

$$E_t \left[\frac{1}{2} \int_t^\infty e^{-\int_t^s \rho_m(a^-) da} \left((\pi_s - \bar{\pi})^2 + \alpha_x x_s^2 + \alpha_r r_s^2 \right) ds \right]$$

subject to the dynamics of inflation and output gaps given by [\(24\)](#).

Proposition 6 *The optimal monetary policy for regime k is the same as that for the single-regime case if we use $-\Lambda_{kk}$ as the monetary discount rate, where Λ_{kk} is the k th eigenvalue of matrix A as defined by equation [\(37\)](#). Thus, the optimal monetary policy rules with switching regimes in the monetary discount rate ρ_m are respectively linear in inflation and output gap in each regime.*

This result greatly simplifies the optimization problem with regime-dependent monetary discount rates. We can simply transform the regime-dependent problem into separate single-regime problem solved in the previous section. Thus, all properties discussed in the single-regime model can be extended into the regime-dependent case. More importantly, the optimal monetary policy rules are also linear when the monetary discount rates switch between regimes. Notice that $Q = 0$ if regime does not switch. In this case, $\Lambda_{kk} = -\rho_{mk}$, thus, [Proposition 6](#) exactly recovers the results for the case of single monetary regime.

3.2 Term Structure of Interest Rates

Since the optimal monetary policy is linear, we discuss the term structure of interest rates under a generic linear interest rate rule that includes the optimal monetary policy as a special case.

3.2.1 Single Monetary Policy Regime

In this economy, the short-term nominal interest rate is set by the central bank based on the optimal interest rate rule (28) and the equilibrium dynamics of the inflation and output gap are also determined by (30). Given the prices of risk, nominal bond prices with all maturities and hence the term structure of (nominal) interest rate can be determined by the optimal monetary policy rule.

Since the optimal interest rule is linear, we present the term structure results based on a generic linear interest rate rule as

$$r_t = \beta_0 + \beta_\pi(\pi_t - \bar{\pi}) + \beta_x x_t, \quad (38)$$

which includes the optimal interest rate rule (28) as a special case.

Given the dynamics of inflation and output gap under the physical measure in (24) and the market price of risk, we obtain the following risk-neutral dynamics of the two macro variables

$$d \begin{pmatrix} \pi_t - \bar{\pi} \\ x_t \end{pmatrix} = \left\{ \theta + \begin{bmatrix} \tilde{\rho} & -\kappa_y \\ \frac{1}{\gamma}(\beta_\pi - 1) & \frac{1}{\gamma}\beta_x \end{bmatrix} \begin{pmatrix} \pi_t - \bar{\pi} \\ x_t \end{pmatrix} \right\} dt + \begin{bmatrix} \sigma_\pi^\top \\ \sigma_x^\top \end{bmatrix} dZ_t^*, \quad (39)$$

where $Z_t^* = Z_t + \lambda_t dt$ is the Brownian motion under the risk-neutral measure, and

$$\theta = \begin{pmatrix} -\sigma_\pi \cdot (\sigma_p + \gamma\sigma_x) \\ \frac{1}{\gamma}[\beta_0 - \tilde{r} - \gamma\sigma_x \cdot (\sigma_p + \gamma\sigma_x)] \end{pmatrix}.$$

Due to the affine structure of both the interest rate rule and the dynamics of the state variables, we easily obtain the following closed-form pricing formula for a zero-coupon bond with time-to-maturity τ ,

$$B(\tau, \pi_t, x_t) = e^{-F(\tau) - G_\pi(\tau)(\pi_t - \bar{\pi}) - G_x(\tau)x_t},$$

where

$$F' - \theta^\top G + \frac{1}{2} \|\sigma_\pi\|^2 G_\pi^2 + \frac{1}{2} \|\sigma_x\|^2 G_x^2 + \sigma_\pi \cdot \sigma_x G_\pi G_x - \beta_0 = 0,$$

$$G' - \begin{bmatrix} \tilde{\rho} & -\kappa_y \\ \frac{1}{\gamma}(\beta_\pi - 1) & \frac{1}{\gamma}\beta_x \end{bmatrix}^\top G - \begin{pmatrix} \beta_\pi \\ \beta_x \end{pmatrix} = \mathbf{0},$$

where $G = (G_\pi, G_x)^\top$, and $F(0) = G_\pi(0) = G_x(0) = 0$ because $B(0, \pi, x) = 1$, bond price is worth par at maturity.

Given the zero bond price at different maturities, the zero yield curve is then

$$Y_t(\tau) = -\frac{\ln B(\tau, \pi_t, x_t)}{\tau} = \frac{F(\tau) + (\pi - \bar{\pi}, x_t)G(\tau)}{\tau}.$$

3.2.2 Multiple Monetary Policy Regimes

When central bank may switch among discount rates in its loss function, the optimal monetary policy also switches among regimes. As shown in Proposition 6, the optimal monetary policy under each regime is similar to the single-regime one after we appropriately adjust the monetary discount rate. Thus, for the monetary policy regime $k = 1, \dots, K$, the monetary policy rule is given by

$$r_{kt} = \beta_{k0} + \beta_{k\pi}(\pi_t - \bar{\pi}) + \beta_{kx}x_t, \quad (40)$$

where β_{k0} , $\beta_{k\pi}$ and β_{kx} are for each k , and the monetary policy regime follows a continuous-time Markov chain with a constant transition rate matrix Q .

The regime-dependent monetary policy results in macro dynamics becoming regime-dependent via aggregate demand. Under regime k , the dynamics of macro variables (39) become

$$d \begin{pmatrix} \pi_t - \bar{\pi} \\ x_t \end{pmatrix} = \left\{ \theta_k + \begin{bmatrix} \tilde{\rho} & -\kappa_y \\ \frac{1}{\gamma}(\beta_{k\pi} - 1) & \frac{1}{\gamma}\beta_{kx} \end{bmatrix} \begin{pmatrix} \pi_t - \bar{\pi} \\ x_t \end{pmatrix} \right\} dt + \sigma dZ_t^*, \quad (41)$$

where

$$\theta_k = \begin{pmatrix} -\sigma_\pi \cdot (\sigma_p + \gamma\sigma_x) \\ \frac{1}{\gamma}[\beta_{k0} - \tilde{r} - \gamma\sigma_x \cdot (\sigma_p + \gamma\sigma_x)] \end{pmatrix}, \quad \sigma = \begin{bmatrix} \sigma_\pi^\top \\ \sigma_x^\top \end{bmatrix}.$$

Thus, monetary policy rule (40) and the dynamics π_t and x_t form a 2-factor affine term structure model with switching regime.

Following the same approximation scheme used in Li et al. (2013), the prices of nominal bonds that pay \$1 at maturity date T are given by

$$e^{-F_k(\tau) - (\pi_t - \bar{\pi}, x_t)G_k(\tau)},$$

under regime k , where $\tau = T - t$, and F_k and G_k satisfy the following ordinary differential equations (ODEs)

$$F'_k - \theta_k^\top G_k + \frac{1}{2} \mathbf{Trace} \left(\sigma^\top G_k G_k^\top \sigma \right) - \beta_{k0} + \sum_{n=1}^K Q_{kn} e^{-F_n + F_k} = 0 \quad (42)$$

$$G'_k - \kappa_k^\top G_k - \sum_{n=1}^K Q_{kn} e^{-F_n + F_k} [G_n - G_k] - \beta_k = \mathbf{0}, \quad (43)$$

with $F_k(0) = 0$ and $G_k(0) = \mathbf{0}$ for all k , where

$$\kappa_k = \begin{bmatrix} \tilde{\rho} & -\kappa_y \\ \frac{1}{\gamma}(\beta_{k\pi} - 1) & \frac{1}{\gamma}\beta_{kx} \end{bmatrix}, \quad \beta_k = \begin{pmatrix} \beta_{k\pi} \\ \beta_{kx} \end{pmatrix}.$$

The bond prices for the case of single regime are obtained by setting $Q_{kn} \equiv 0$.

4 Empirical Analysis

We apply our model to analyze the US monetary policy using both macro and term structure data. One of the important empirical question we try to address is that whether monetary policy in the past is optimal or not relative to our model and what are the implications of the estimated monetary policy for macro stability. In this regard, our empirical studies are investigating the past monetary policy through the lens of our theoretical model. In a reduced-form, no-arbitrage macro

term structure model, [Li et al. \(2013\)](#) show that there were two distinct monetary policy regimes in the US from 1952 to 2007. One of them satisfies the *Taylor principle*, and the other does not. This result confirms the arguments presented in [Taylor \(1993\)](#). Thus, we estimate a two-regime version of our model because a single monetary policy rule may not be a good representation what happened in the past. As shown in the numerical examples in the last section, different monetary discount rates can lead to quite different policy rules in terms of the coefficients of the inflation and output gap, and hence different macro dynamics. We use two regimes to represent roughly how the Fed potential weighs between near-term (large discount rate) and long-term (small discount rate) macro fluctuations.

4.1 Specification and Estimation Method

Since not all of the structural parameters are identifiable in our empirical exercise, we preset the following parameters: $\sigma_a = 0$, $\sigma_p = 0$, $\sigma_\pi = (\sigma_{\pi\pi}, 0)^\top$, and $\sigma_x = (\sigma_{\pi x}, \sigma_{xx})^\top$. We also estimate the coefficients of monetary policy directly and examine whether the estimated coefficients are optimal for any possible monetary objective afterwards.

We use the MCMC method in [Li et al. \(2013\)](#) to estimate the model. The two state variables, inflation rate π_t and output gap x_t , satisfy

$$d \begin{pmatrix} \pi_t \\ x_t \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ \frac{\beta_{k0} - \bar{r}}{\gamma} \end{pmatrix} + \kappa_k \begin{pmatrix} \pi_t - \bar{\pi} \\ x_t \end{pmatrix} \right\} dt + \sigma dZ_t \quad (44)$$

under the physical measure. Thus, the likelihood function of $(\pi_t - \bar{\pi}, x_t)$ can be derived.

In addition to the state variables, we also use bond yields to estimate the model. Given a time series of realized yields for M zero-coupon bonds with a maturity τ_m , $\hat{Y}_t(\tau_m)$, for $t = 1, \dots, T$ and $m = 1, \dots, M$, we assume that the differences between the model yields and their empirical counterparts satisfy iid normal across maturities, that is

$$\hat{Y}_t(\tau_m) = Y_{k_t, t}(\tau_m) + \epsilon_{m, t}, \quad \epsilon_{m, t} \sim N(0, \sigma_m), \quad (45)$$

where $Y_{k,t}(\tau_m)$ is the model yield for monetary regime k_t , given by

$$Y_{k,t}(\tau_m) = \frac{F_k(\tau_m) + (\pi_t - \bar{\pi}, x_t)G_k(\tau_m)}{\tau_m}, \quad (46)$$

where F_k and G_k solve ODEs (42) and (43), respectively.

Note that a key difference between this model and the one in Li et al. (2013) is the dynamics of inflation and output gap. The dynamics are exogenously specified and independent of the Taylor rule used in Li et al. (2013), whereas they are endogenously derived in a general equilibrium model with sticky price and consistent with a linear nominal interest rate rule.

4.2 Data

The data used in our empirical analysis consists of zero coupon yields of US Treasury bonds with maturities of 3 months, 1, 2, 3, 4, and 5 years as well as GDP and GDP deflator observed at quarterly frequency between the second quarter of 1952 and the third quarter of 2007. The US Treasury yields are the Fama-Bliss yields available in the CRSP database, and the GDP and GDP deflator are downloaded from the website of St. Louis Fed. The output gap is obtained by applying the HP filter to the quarterly GDP data.

4.3 Empirical Results

4.3.1 Single Monetary Policy Regime

We first estimate the model with single monetary policy regime. The parameter estimates and their standard errors are reported in Table 1.

The estimated monetary policy rule is

$$r_t = \underset{(0.0007)}{0.0537} + \underset{(0.0286)}{0.8036} (\pi_t - 0.03) + \underset{(0.0304)}{0.0343} x_t. \quad (47)$$

Two interesting features of the estimated monetary policy rule stand out. One is that the coefficient of output gap is positive, although not statistically significant. This indicates that the estimated monetary policy rule cannot be optimal from the perspective of our model. The other is that

Table 1: **Parameter Estimates**

This table reports the empirical estimates of the model parameters for the single-regime model. We run MCMC with 100,000 iterations and use the posterior mean (standard deviation) of the last 50,000 iterations as estimates of the model parameters (standard error, shown in parentheses).

| Parameter | Single Regime | | Two Regimes | | | |
|------------------------|---------------|----------|-------------|----------|----------|----------|
| | - | | Regime 1 | | Regime 2 | |
| ρ | 0.0159 | (0.0007) | 0.0168 | | (0.0016) | |
| γ | 0.5808 | (0.1502) | 3.7910 | | (0.5369) | |
| $\nu\mu_a$ | 0.0148 | (0.0017) | 0.0136 | | (0.0006) | |
| κ_y | 0.0104 | (0.0105) | 0.0276 | | (0.0197) | |
| $\sigma_{\pi\pi}$ | 0.0080 | (0.0004) | 0.0083 | | (0.0004) | |
| σ_{xx} | 0.0176 | (0.0008) | 0.0192 | | (0.0009) | |
| $\sigma_{x\pi}$ | -0.0015 | (0.0007) | -0.0047 | | (0.0010) | |
| β_0 | 0.0537 | (0.0006) | 0.0756 | (0.0010) | 0.0398 | (0.0009) |
| β_π | 0.8036 | (0.0286) | 1.1350 | (0.0767) | 0.6511 | (0.0225) |
| β_x | 0.0343 | (0.0304) | -0.1650 | (0.0743) | 0.2772 | (0.0373) |
| Q_{kk} | - | - | -0.0184 | (0.0080) | -0.1433 | (0.0248) |
| $\sigma_{m=1,\dots,6}$ | 0.0217 | (0.0011) | 0.0134 | | (0.0007) | |
| | 0.0213 | (0.0010) | 0.0122 | | (0.0006) | |
| | 0.0214 | (0.0010) | 0.0115 | | (0.0006) | |
| | 0.0213 | (0.0010) | 0.0116 | | (0.0006) | |
| | 0.0215 | (0.0010) | 0.0120 | | (0.0006) | |
| | 0.0215 | (0.0010) | 0.0122 | | (0.0006) | |

the coefficient of inflation is less than 1. According to Proposition 2, these two features of the monetary policy rule imply that the dynamics of the two macro variables do not follow a stable process. The nature of macro stability can be further illuminated by examining the estimated dynamics of inflation and output gap:

$$d \begin{pmatrix} \pi_t \\ x_t \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ -0.0007 \\ (0.0027) \end{pmatrix} + \begin{bmatrix} 0.0096 & -0.0104 \\ (0.0027) & (0.0105) \end{bmatrix} \begin{pmatrix} \pi_t - 0.03 \\ x_t \end{pmatrix} \right\} dt + \begin{bmatrix} 0.0080 & 0 \\ (0.0004) & \\ -0.0015 & 0.0176 \\ (0.0007) & (0.0008) \end{bmatrix} dZ_t.$$

Intuitively, for the above linear system to be stable, it has to be mean-reverting. However, the parameter estimates show that the two eigenvalues for the estimated κ -matrix are (-0.0292, 0.1037), one of which is positive. Thus, instead of converging to inflation target and zero output gap, this system tends to explode. This suggests that it is very unlikely that US had one single regime of monetary policy during the sample period. As shown below, it is also evident from the model's performance in explaining the term structure of interest rates.

To assess how well the model explains the observed yields, we regress the observed yields on

Table 2: Regression Analysis of Observed Yields on Model Yields

This table provides the regression analysis of observed zero-coupon government bond yields on model-implied yields under the single-regime model at different maturities. The regression equation is

$$\text{Observed Yields} = \gamma_0 + \gamma_1 \text{Model Yields} + \text{error},$$

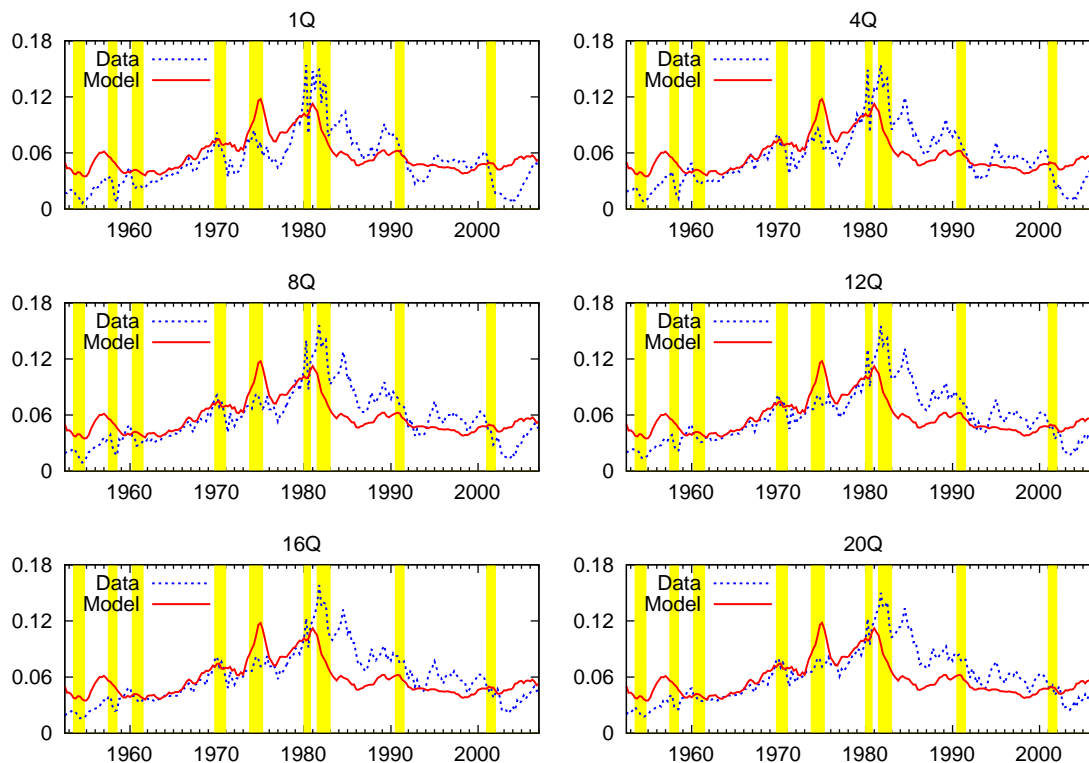
where the model yields are computed based on estimated parameters in the previous table. Standard errors are reported in parentheses. The maturities of the bonds range from one quarter (1Q) to five years (20Q).

| Bond Maturity | Single Regime | | | | Two Regimes | | | |
|---------------|---------------------|--------------------|---------------------------------|-------|---------------------|--------------------|---------------------------------|-------|
| | γ_0 | γ_1 | Standard Deviation of Residuals | R^2 | γ_0 | γ_1 | Standard Deviation of Residuals | R^2 |
| 1Q | -0.0102 (0.0046) | 1.0635 (0.0760) | 0.0208 | 47.3% | -0.0087 (0.0022) | 1.0768 (0.0374) | 0.0131 | 79.2% |
| 4Q | -0.0052 (0.0047) | 1.0472 (0.0777) | 0.0213 | 45.5% | -0.0056 (0.0021) | 1.0839 (0.0348) | 0.0123 | 81.7% |
| 8Q | -0.0009 (0.0048) | 1.0061 (0.0786) | 0.0215 | 42.9% | -0.0035 (0.0020) | 1.0666 (0.0320) | 0.0115 | 83.6% |
| 12Q | 0.0039 (0.0047) | 0.9531 (0.0784) | 0.0214 | 40.4% | -0.0002 (0.0020) | 1.0213 (0.0314) | 0.0115 | 82.9% |
| 16Q | 0.0067 (0.0048) | 0.9261 (0.0786) | 0.0215 | 38.9% | 0.0018 (0.0020) | 0.9909 (0.0316) | 0.0117 | 81.9% |
| 20Q | 0.0091 (0.0047) | 0.8999 (0.0778) | 0.0213 | 38.1% | 0.0037 (0.0021) | 0.9534 (0.0315) | 0.0119 | 80.7% |
| mean | 0.0006 | 0.9827 | 0.0213 | 42.2% | -0.0021 | 1.0322 | 0.0120 | 81.7% |

the model implied yields, the results are reported in Table 2. The standard deviations across maturities are almost identical and the constant terms are insignificant from zero. This indicates the model captures the average yields well. However, the regression coefficients are slightly decreasing with maturities, indicating that the single regime model has certain difficulties in capturing long-term yields in the data. This becomes more evident when we examine the R^2 s of these regressions. The R^2 s monotonically decrease from 47% for 3-month yields to 38% for 5-year yields. This shows that the single regime model fails to capture the term structure observed in the Treasury bond markets.

The failure of the single-regime model is also evident when we examine how the model implied yields fair with observed yields over time. Figure 4 plots the model implied yields and the observed yields for different maturities. It shows that the mismatches between the model yields and the observed yields follow certain persistent patterns across maturities. For example, during the 1950s, the model implied yields are consistently higher than the data yields, and during

Figure 4: Data and Model-Implied Yields under the Single-Regime Model



This figure provides the time series plots of observed yields on zero-coupon government bonds at six different maturities and model-implied yields under the single-regime model. Model yields are calculated given the estimated model parameters using the Bayesian MCMC methods. The shaded areas represent the NBER recessions.

the “Great Moderation,” the model implied long-term yields are consistently below the observed long-term yields.

Both the estimated monetary policy rule and term structure performance of the model suggests that the single-regime model does not capture the data well. This is consistent with the results of a no-arbitrage model studied in [Li et al. \(2013\)](#).

4.3.2 Two Monetary Policy Regimes

The parameter estimates for the model with two monetary policy regimes are also reported in [Table 1](#).

The two estimated monetary policy regimes are

$$r_t = \begin{cases} 0.0756 + 1.1350 (\pi_t - 0.03) - 0.1650 x_t, & \text{Regime 1,} \\ (0.0010) \quad (0.0767) \quad (0.0743) \\ 0.0398 + 0.6511 (\pi_t - 0.03) + 0.2772 x_t, & \text{Regime 2.} \\ (0.0009) \quad (0.0225) \quad (0.0373) \end{cases} \quad (48)$$

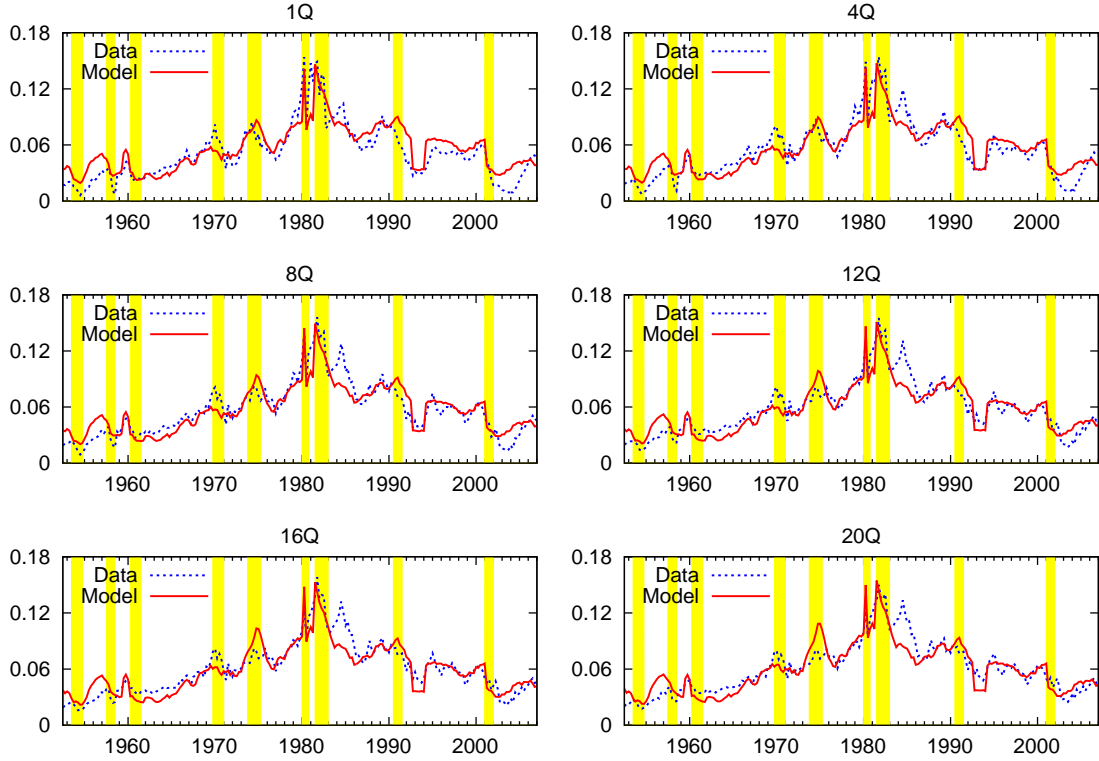
One obvious observation is that the two monetary policy rules are quite different from each other. More importantly, the distinctions between the two regimes have opposite implications for the economy, as shown in the theoretical model. Under regime 1, the coefficient of the monetary policy on output gap is significantly negative,¹² thus, the monetary policy under regime 1 satisfies one of the necessary conditions for the policy to be optimal: negative coefficient on output gap. Moreover, the coefficient on inflation is greater than 1 and statistically significant. On the other hand, the monetary policy under regime 2 cannot be optimal similarly to the case of the single monetary policy regime: the coefficient of output gap is positive. It also cannot achieve macro stability because it violates the necessary condition for the macro variables to be stable set in Proposition 2.

Similar to the single-regime case, we also regress the observed yields on the model implied yields for the two-regime model across maturities; the results are reported in Table 2. The constant terms of these regressions are insignificant and the regression coefficients are close to 1. The standard errors of these regressions are significantly reduced than that for the single-regime model. This indicates the model with two monetary regimes is a more probable description of the US monetary policy over the sample period. This becomes more evident when we examine the regression R^2 s. The average R^2 is increased roughly from 42% of the single-regime model to 82% of the two-regime model. Unlike the single-regime model, the R^2 s of the two-regime model are roughly even across maturities. Furthermore, Figure 5 also shows the model implied yields of the two-regime model fit the observed yields much better dynamically than that of the single regime model.

Now we examine macro stability under different monetary policy regimes. As indicated in the estimated monetary policy rules, the coefficient of output gap is negative and the coefficient of

¹²This appears conflicting with the results in Li et al. (2013). However, the monetary policy in Li et al. (2013) is based on Taylor rule, in which the coefficient on the output gap is positive.

Figure 5: Data and Model-Implied Yields under the Two-Regime Model



This figure provides the time series plots of observed yields on zero-coupon government bonds at six different maturities and model-implied yields under the two-regime model, where monetary policies are regime-dependent. Model yields are calculated given the estimated model parameters using the Bayesian MCMC methods. The shaded areas represent the NBER recessions.

inflation is greater than 1 under regime 1. The resulting macro dynamics under regime 1 are:

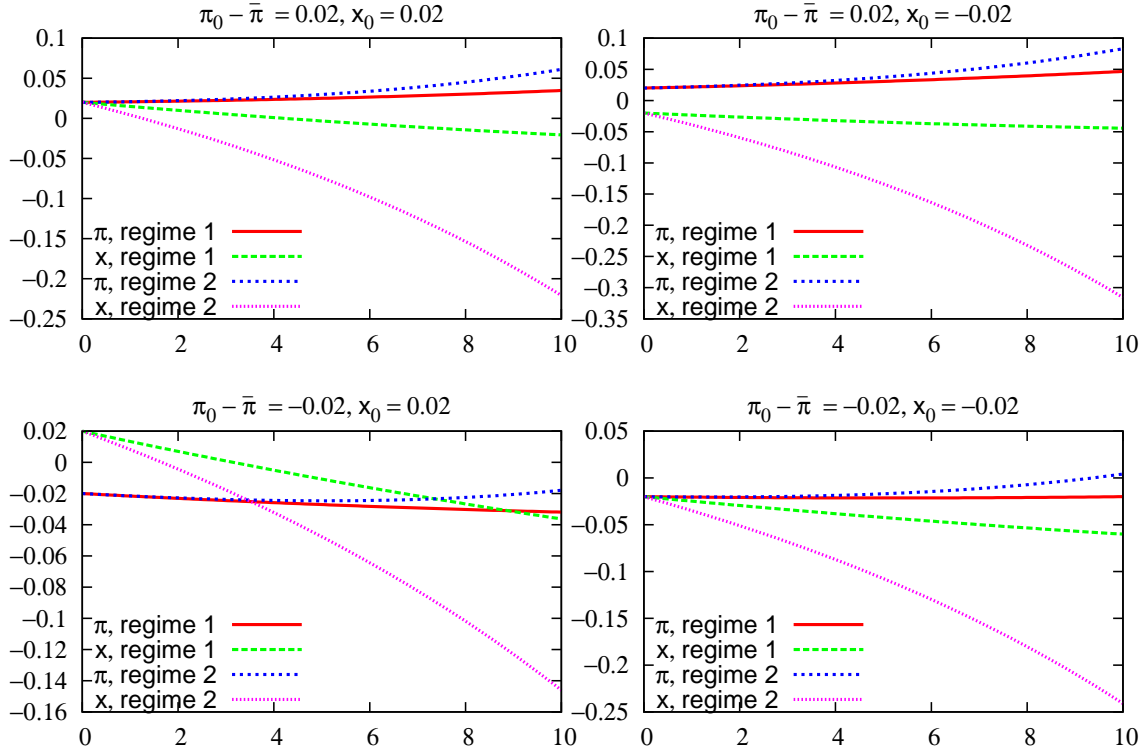
$$d \begin{pmatrix} \pi_t \\ x_t \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ -0.0051 \\ (0.0012) \end{pmatrix} + \begin{bmatrix} 0.0548 & -0.0276 \\ (0.0072) & (0.0197) \\ 0.0364 & -0.0447 \\ (0.0219) & (0.0220) \end{bmatrix} \begin{pmatrix} \pi_t - 0.03 \\ x_t \end{pmatrix} \right\} dt + \begin{bmatrix} 0.0083 & 0 \\ (0.0004) & \\ -0.0047 & 0.0192 \\ (0.0010) & (0.0009) \end{bmatrix} dZ_t.$$

Under the monetary policy regime 2, the coefficient of output gap is positive and the coefficient of inflation is less than 1. The resulting macro dynamics are:

$$d \begin{pmatrix} \pi_t \\ x_t \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ -0.0148 \\ (0.0009) \end{pmatrix} + \begin{bmatrix} 0.0548 & -0.0276 \\ (0.0072) & (0.0197) \\ -0.0937 & 0.0747 \\ (0.0136) & (0.0151) \end{bmatrix} \begin{pmatrix} \pi_t - 0.03 \\ x_t \end{pmatrix} \right\} dt + \begin{bmatrix} 0.0083 & 0 \\ (0.0004) & \\ -0.0047 & 0.0192 \\ (0.0010) & (0.0009) \end{bmatrix} dZ_t.$$

Even though the local volatility matrix σ are the same across policy regimes, the long-term dynamics of inflation and output gaps behave quite differently depending on the property of the κ -matrix, which is directly determined by the monetary policy.

Figure 6: Expected Future Gaps

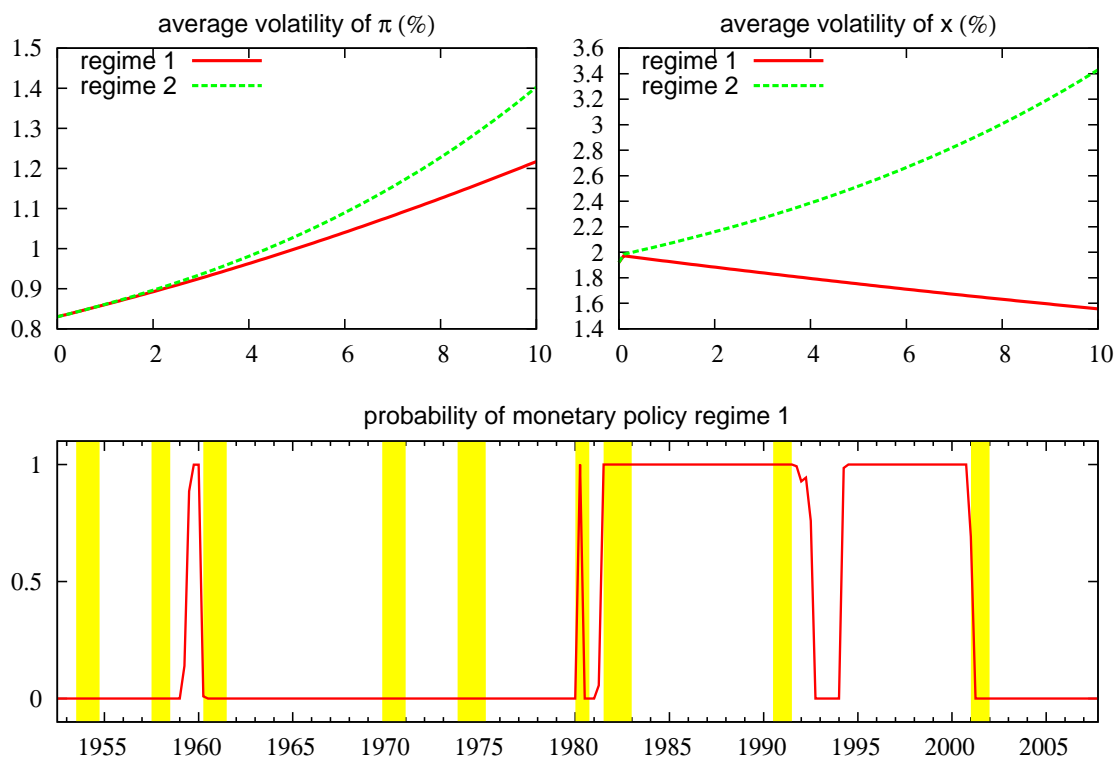


This figure provides the expected future inflation and output gaps calculated by equation (31) in the two-regime model.

The eigenvalues of the κ -matrix are (0.0434, -0.0333) under regime 1 and (0.0129, 0.1166) under regime 2. Thus, the system of the macro variables is not absolutely stable because some eigenvalues are positive. It requires all eigenvalues being negative for the system to be absolutely stable. However, we can assess the relative macro stability between the two monetary regimes through the expected future inflation and output gaps and their average variance or volatility, which are given by equations (31) and (32) respectively.

Figure 6 plots the expected future inflation and output gap for four cases of starting values. A striking feature is that the expected output gap is drifting lower for all cases. Two reasons cause this downward output gap. One is that the κ -matrix lacks the mean-reverting force due to certain eigenvalues being positive. The other is that β_0 is relatively lower, and thus, θ^P is negative for the output gap. However, the downward slope is much smaller under policy regime 1, in which β_0 is much higher and the κ -matrix has one negative eigenvalue.

Figure 7: Average Volatility of Gaps and Estimated Regimes



This figure provides the average volatility of inflation and output gaps calculated by equation (32) in the two-regime model in the upper panel and the estimated probability of policy regime 1 in the lower panel.

The benefit of policy regime 1 becomes more evident by the average volatility of the output gap as shown in Figure 7. Even though the average volatility of inflation is similar between the two policy regimes, the average volatility of output gap behaves very differently. The average volatility of output gap is becoming lower over longer time period under policy regime 1, whereas it becomes progressively higher under policy regime 2. Thus, the output gap is stable under policy regime 1 and unstable under policy regime 2. This implication makes more sense when we examine the estimated historical policy regime that is reported in the lower panel of Figure 7. The estimated policy regime implies that the Fed undertook regime 1 policy roughly over the period from 1981 to 2001. Combining this with the “Great Moderation,” which Economists refer to the period from 1985 to 2006, implies that the relatively optimal monetary policy plays a decisive role in stabilizing the output gap. It is also very intriguing to notice that the “Great Moderation” started after the optimal policy being effective for about 2 to 3 years and it ended by the “Great

Recession” after policy regime 1 had ended for about 4 to 5 years. As suggested by the top two panels in Figure 7, it takes time to notice the results of policy rules regardless it is optimal or not. This makes perfect sense since the policy rules are systematic ones.

Table 3: **Fitted Parameters of Monetary Loss Function and Implied Optimal Policy Coefficients**

| ρ_m | α_x | α_r | β_0 | β_π | β_x |
|----------|------------|------------|----------------|----------------|------------------|
| 0.197 | 0.000 | 0.979 | 0.043 | 1.120 | -0.238 |
| | | | [0.074, 0.078] | [0.985, 1.285] | [-0.311, -0.019] |

The optimal policy coefficients β s are based on single-regime model. The parentheses are the 95% confidence intervals of the estimates under regime 1 in the two-regime model.

Even though the monetary policy under regime 1 is much better than the one under regime 2, it is far from perfect in the sense that it does not achieve absolute stability. To further examine the optimality of the monetary policy under regime 1, we fit the parameters of the loss function to minimize the squared differences between the implied monetary policy and the estimated policy under regime 1. The fitted parameters and the associated optimal monetary policy are reported in Table 3. The fitted monetary discount rate, 0.197, is very large, and the coefficient for the output gap is very close to 0. Although the coefficients on inflation and output gaps are within the 95% confidence intervals of the estimated counterparts, the resulting constant coefficient of the optimal monetary policy is much lower than the estimated one. Even though we may accept that the estimated policy is close to being optimal, the large monetary discount rate indicates that the Fed puts relatively large weights for near-term variations. The absolute coefficients on inflation and output gaps of the policy rule would be much larger if the Fed had a more balanced weighting scheme on the monetary loss function across time, that is a much lower monetary discount rate.

5 Conclusion

We have developed a continuous-time New Keynesian DSGE model, in which the optimal monetary policy is a simple interest rate rule that is a linear function of inflation and output gaps. This optimal interest rate rule and the equilibrium dynamics of inflation and output gaps jointly form an affine macro term structure model, in which bond yields are directly tied to the macroeconomic variables. This enables us to use the yield data to identify the past US monetary policies and the

dynamics of the macroeconomic variables. The empirical results suggest that there are two distinct systematic monetary policy regimes in the past. Although none of the two policy regimes can achieve absolute macroeconomic stability, the near optimal one is much more stabilizing than the non-optimal one.

The continuous-time setup offers some new perspectives and insights to one of the most important questions faced by economists and policy makers: how to conduct monetary policy and what are the long-term impacts of such policies to the economy. The continuous-time technique is well-developed and widely used in Finance to model uncertainty, e.g., portfolio choice, consumption-based asset pricing, and term structure models. This paper demonstrates that it can also be a powerful tool to analyze issues in macroeconomics.

In this paper, the growth rate of output, and hence the real interest rate in the flexible price equilibrium, is constant. We can extend the base model by adding a time-varying, stochastic growth rate of technology. Another direction to extend the model is to incorporate preference shocks like the habit-formation of [Campbell and Cochrane \(1999\)](#) to study monetary policy risk implications ([Campbell et al., 2014](#)). Thus, price of risk is time-varying and stochastic; time-varying price of risk is very important to explain the observed behaviors of asset prices.

Appendix

In this appendix, we provide technical details on certain analyses in the paper. In Section A.1, we discuss the log-linear approximation used in deriving the sticky price equilibrium. In Section A.2, we discuss how to solve the cost function used in determining optimal monetary policy. In Section A.3, we provide the proofs of the propositions in the main text.

A.1 Log-Linear Approximations

Using the following approximations

$$\begin{aligned} C_s^{-1-\gamma} &= \bar{C}_s^{-1-\gamma} \left(\frac{C_s}{\bar{C}_s} \right)^{1-\gamma} \approx \bar{C}_s^{-1-\gamma} \left(1 + (1-\gamma) \ln \frac{C_s}{\bar{C}_s} \right), \\ \left(\frac{P_s}{P_t e^{\bar{\pi}(s-t)}} \right)^\epsilon &\approx 1 + \epsilon \ln \frac{P_s}{P_t e^{\bar{\pi}(s-t)}}, \\ \frac{\varphi_s}{\bar{\varphi}} &\approx 1 + \ln \frac{\varphi_s}{\bar{\varphi}}, \end{aligned}$$

where $\bar{C}_t = \bar{Y}_t$ is flexible-price equilibrium consumption. Then we have

$$\begin{aligned} &E_t \left[\int_t^\infty e^{-(\delta+\rho)(s-t)} C_s^{1-\gamma} \frac{\varphi_s}{\bar{\varphi}} \left(\frac{P_s}{P_t e^{\bar{\pi}(s-t)}} \right)^\epsilon ds \right] \\ &\approx E_t \left[\int_t^\infty e^{-(\delta+\rho)(s-t)} \bar{C}_s^{1-\gamma} \left(1 + (1-\gamma) \ln \frac{C_s}{\bar{C}_s} + \epsilon \ln \frac{P_s}{P_t e^{\bar{\pi}(s-t)}} + \ln \frac{\varphi_s}{\bar{\varphi}} \right) ds \right], \\ &E_t \left[\int_t^\infty e^{-(\delta+\rho)(s-t)} C_s^{1-\gamma} \left(\frac{P_s}{P_t e^{\bar{\pi}(s-t)}} \right)^{\epsilon-1} ds \right] \\ &\approx E_t \left[\int_t^\infty e^{-(\delta+\rho)(s-t)} \bar{C}_s^{1-\gamma} \left(1 + (1-\gamma) \ln \frac{C_s}{\bar{C}_s} + (\epsilon-1) \ln \frac{P_s}{P_t e^{\bar{\pi}(s-t)}} \right) ds \right]. \end{aligned}$$

Thus, by equation (13), we have

$$\begin{aligned} \frac{P_t^* - P_t}{P_t} &\approx \frac{E_t \left[\int_t^\infty e^{-(\delta+\rho)(s-t)} \bar{C}_s^{1-\gamma} \left(\ln \frac{P_s}{P_t e^{\bar{\pi}(s-t)}} + \ln \frac{\varphi_s}{\bar{\varphi}} \right) ds \right]}{E_t \left[\int_t^\infty e^{-(\delta+\rho)(s-t)} \bar{C}_s^{1-\gamma} \left(1 + (1-\gamma) \ln \frac{C_s}{\bar{C}_s} + (\epsilon-1) \ln \frac{P_s}{P_t e^{\bar{\pi}(s-t)}} \right) ds \right]} \\ &\approx \frac{E_t \left[\int_t^\infty e^{-(\delta+\rho)(s-t)} \bar{C}_s^{1-\gamma} \left(\ln \frac{P_s}{P_t e^{\bar{\pi}(s-t)}} + \ln \frac{\varphi_s}{\bar{\varphi}} \right) ds \right]}{E_t \left[\int_t^\infty e^{-(\delta+\rho)(s-t)} \bar{C}_s^{1-\gamma} ds \right]}. \end{aligned} \tag{49}$$

From the results of flexible-price equilibrium, we have

$$\frac{\bar{C}_s}{\bar{C}_t} = \exp \left[\int_t^s v \left(\mu_a - \frac{1}{2} \|\sigma_a\|^2 \right) du + \int_t^s v \sigma_a \cdot dZ_u \right].$$

This implies

$$E_t \left(e^{-(\delta+\rho)(s-t)} \bar{C}_s^{1-\gamma} \right) = \bar{C}_t^{1-\gamma} e^{-(\delta+\tilde{\rho})(s-t)},$$

where $\tilde{\rho}$ is given by equation (16), and

$$E_t \left[\int_t^\infty e^{-(\delta+\rho)(s-t)} \bar{C}_s^{1-\gamma} ds \right] = \frac{\bar{C}_t^{1-\gamma}}{\delta + \tilde{\rho}}.$$

Due to the assumption that the real and nominal shocks are independent, we can separate the expectations with respect to the real variable \bar{C}_s and the nominal variables P_s and φ_s , and hence substituting these two equations back into equation (49) yields equation (17).

A.2 Cost Function

A.2.1 Solving the Monetary Cost Function: Single Regime

Substituting r^* and the derivatives of V into the HJB equation (26) and setting all of the coefficients of the resulting equation equal to zero yield 6 equations. The three equations for three quadratic terms are:

$$(\pi - \bar{\pi})^2 : \quad -\frac{1}{2} \rho_m A_\pi + \tilde{\rho} A_\pi - \frac{1}{2} \frac{1}{\gamma^2 \alpha_r} B_{\pi x}^2 - \frac{1}{\gamma} B_{\pi x} + \frac{1}{2} = 0, \quad (50)$$

$$(\pi - \bar{\pi})x : \quad -\rho_m B_{\pi x} - \kappa_y A_\pi + \tilde{\rho} B_{\pi x} - \frac{1}{\gamma^2 \alpha_r} A_x B_{\pi x} - \frac{1}{\gamma} A_x = 0, \quad (51)$$

$$x^2 : \quad -\frac{1}{2} \rho_m A_x - \kappa_y B_{\pi x} - \frac{1}{2} \frac{1}{\gamma^2 \alpha_r} A_x^2 + \frac{1}{2} \alpha_x = 0. \quad (52)$$

Although there are 6 unknowns to be determined, the first 3 unknowns, A_π , $B_{\pi x}$ and A_x , form a closed system of equations, so we can solve them first. This property simplifies the problem.

If $\rho_m = 2\tilde{\rho}$, then equation (51) shows $B_{\pi x} < 0$, and hence equation (50) implies

$$B_{\pi x} = -\gamma \alpha_r \left(1 + \sqrt{1 + \frac{1}{\alpha_r}} \right).$$

Then we can solve A_x by equation (52) and A_π by equation (51).

If $\rho_m \neq 2\tilde{\rho}$, let

$$z \equiv \frac{1}{\gamma\alpha_r} B_{\pi x} + 1, \quad \text{and hence} \quad B_{\pi x} = \gamma\alpha_r(z - 1). \quad (53)$$

Thus, equation (50) implies

$$A_\pi = \frac{\alpha_r z^2 - (1 + \alpha_r)}{2\tilde{\rho} - \rho_m}, \quad (54)$$

and thus, substituting this into equation (51) implies

$$\begin{aligned} A_x &= \frac{\gamma^2 \alpha_r (\tilde{\rho} - \rho_m)(z - 1)}{z} - \gamma \frac{\kappa_y \alpha_r z^2 - \kappa_y (1 + \alpha_r)}{(2\tilde{\rho} - \rho_m)z} \\ &= \frac{1}{z} \left[-\frac{\gamma \kappa_y \alpha_r}{2\tilde{\rho} - \rho_m} z^2 + \gamma^2 \alpha_r (\tilde{\rho} - \rho_m) z + \left(\frac{\gamma \kappa_y (1 + \alpha_r)}{2\tilde{\rho} - \rho_m} - \gamma^2 \alpha_r (\tilde{\rho} - \rho_m) \right) \right] \\ &\equiv \frac{1}{z} (-G_2 z^2 + G_1 z + G_0), \end{aligned} \quad (55)$$

where we use G s to simplify the coefficients. Substituting this equation and the expression for $B_{\pi x}$ into equation (52) yields a quartic equation

$$\begin{aligned} \rho_m (-G_2 z^2 + G_1 z + G_0) z + 2\kappa_y \gamma \alpha_r (z - 1) z^2 + \frac{1}{\gamma^2 \alpha_r} (-G_2 z^2 + G_1 z + G_0)^2 - \alpha_x z^2 \\ = \frac{G_2^2}{\gamma^2 \alpha_r} z^4 + \left(2\gamma \kappa_y \alpha_r - \rho_m G_2 - \frac{2G_1 G_2}{\gamma^2 \alpha_r} \right) z^3 \\ + \left(\rho_m G_1 - 2\gamma \kappa_y \alpha_r + \frac{G_1^2 - 2G_2 G_0}{\gamma^2 \alpha_r} - \alpha_x \right) z^2 + \left(\rho_m G_0 + \frac{2G_1 G_0}{\gamma^2 \alpha_r} \right) z + \frac{G_0^2}{\gamma^2 \alpha_r} = 0. \end{aligned}$$

Rewrite this quartic equation explicitly as:

$$\begin{aligned} \frac{\kappa_y^2 \alpha_r^2}{(2\tilde{\rho} - \rho_m)^2} z^4 + \gamma \kappa_y \alpha_r^2 z^3 - \alpha_r \left(2\gamma \kappa_y \alpha_r + \frac{2\kappa_y^2 (1 + \alpha_r)}{(2\tilde{\rho} - \rho_m)^2} - \gamma^2 \alpha_r \tilde{\rho} (\tilde{\rho} - \rho_m) + \alpha_x \right) z^2 \\ + \alpha_r \left(\gamma \kappa_y (1 + \alpha_r) - \gamma^2 \alpha_r (\tilde{\rho} - \rho_m) (2\tilde{\rho} - \rho_m) \right) z \\ + \left(\frac{\kappa_y (1 + \alpha_r)}{2\tilde{\rho} - \rho_m} - \gamma \alpha_r (\tilde{\rho} - \rho_m) \right)^2 = 0. \end{aligned} \quad (56)$$

We solve this equation for z and hence we obtain solutions for A_π , $B_{\pi x}$, and A_x . This equation has four solutions. However, our numerical experiments show that only one solution yields both A_π

and A_x being positive, which are necessary conditions for optimality.

The coefficients of linear terms of the state variables show

$$\pi - \bar{\pi} : -\rho_m C_\pi + \tilde{\rho} C_\pi - \frac{1}{\gamma^2 \alpha_r} B_{\pi x} C_x - \frac{1}{\gamma} \tilde{r} B_{\pi x} - \frac{1}{\gamma} C_x = 0, \quad (57)$$

$$x : -\rho_m C_x - \kappa_y C_\pi - \frac{1}{\gamma^2 \alpha_r} A_x C_x - \frac{1}{\gamma} \tilde{r} A_x = 0, \quad (58)$$

where \tilde{r} is defined by (23). Given A_π , A_x and $B_{\pi x}$, these two equations solve C_x and C_π as follows:

$$\begin{pmatrix} C_x \\ C_\pi \end{pmatrix} = -\tilde{r} \begin{bmatrix} \gamma \rho_m + \frac{1}{\gamma \alpha_r} A_x & \gamma \kappa_y \\ z & \gamma(\rho_m - \tilde{\rho}) \end{bmatrix}^{-1} \begin{pmatrix} A_x \\ B_{\pi x} \end{pmatrix}. \quad (59)$$

Last, the constant term D is solved through the following equation

$$-\rho_m D - \frac{1}{2} \frac{1}{\gamma^2 \alpha_r} C_x^2 - \tilde{r} C_x + \frac{1}{2} \|\sigma_\pi\|^2 A_\pi + \frac{1}{2} \|\sigma_x\|^2 A_x + \sigma_x \cdot \sigma_\pi B_{\pi x} = 0. \quad (60)$$

Thus, we have the solution to the cost function, and hence the optimal monetary policy.

Note that our numerical experiments also confirm that the resulting cost function is well behaved. It is always positive and satisfies transversality condition, $\lim_{t \rightarrow \infty} e^{-\rho_m t} E[V(\pi_t, x_t)] \rightarrow 0$.

A.2.2 Global Minimum of Monetary Cost Function

By the nature of the monetary cost function, $V(\pi, x)$ has to be globally positive, and thus, it has to be globally convex and bounded. That is $A_\pi > 0$, $A_x > 0$, and $A_\pi A_x > B_{\pi x}^2$. While it would be very tedious, if not impossible, to show global convexity in general, it is straightforward to check this numerically. Our numerical experiments reveal that it seems that $A_\pi A_x > B_{\pi x}^2$ always holds when both A_π and A_x are positive.

The monetary cost function achieves its global minimum at

$$\pi^* - \bar{\pi} = \frac{B_{\pi x} C_x - A_x C_\pi}{A_\pi A_x - B_{\pi x}^2}, \quad x^* = \frac{B_{\pi x} C_\pi - A_\pi C_x}{A_\pi A_x - B_{\pi x}^2},$$

and the minimum is given by

$$\begin{aligned} \min_{\pi, x} V(\pi, x) = V(\pi^*, x^*) = & -\frac{1}{2} \frac{A_\pi C_x^2 + A_x C_\pi^2 - 2B_{\pi x} C_x C_\pi}{A_\pi A_x - B_{\pi x}^2} - \frac{1}{\rho_m} \left(\frac{1}{2} \frac{1}{\gamma^2 \alpha_r} C_x^2 + \tilde{r} C_x \right) \\ & + \frac{1}{\rho_m} \left(\frac{1}{2} \|\sigma_\pi\|^2 A_\pi + \frac{1}{2} \|\sigma_x\|^2 A_x + \sigma_x \cdot \sigma_\pi B_{\pi x} \right). \end{aligned} \quad (61)$$

Notice that the first two terms in the minimum are proportional to \tilde{r}^2 because C_x and C_π are proportional to \tilde{r} . The last term is always positive and independent of \tilde{r} . This also means that the summation of the first two terms are greater than or equal to zero.

A.2.3 Zero Monetary Control Cost

In the optimal monetary policy literature, it is common to take the monetary cost, α_r , to be zero. If $\alpha_r = 0$, then the optimization problem degenerates: the loss function changes from quadratic in control into linear in control. Without the quadratic term in nominal interest rate r , the solution to the HJB (26) would be like $r^* = -\infty$ either $\frac{\partial V}{\partial x} > 0$ and $r^* = \infty$ if $\frac{\partial V}{\partial x} < 0$ or $\frac{\partial V}{\partial x} \equiv 0$. The later case means that V does not depend on output gap x . However, the solution does not yield explicit controls. The asymptotic method offers not only explicit control but also an approximation solution to the control problem by using an arbitrary small α_r .

Closely examining equation (56) reveals that the only viable solution involves the convergence of $\alpha_r z^2$ as α_r goes to zero. In this case, equation (56) becomes

$$\begin{aligned} & \frac{\kappa_y^2}{(2\tilde{\rho} - \rho_m)^2} (\alpha_r z^2)^2 - \left(\frac{2\kappa_y^2}{(2\tilde{\rho} - \rho_m)^2} + \alpha_x \right) (\alpha_r z^2) + \frac{\kappa_y^2}{(2\tilde{\rho} - \rho_m)^2} = 0 \\ \Leftrightarrow & \frac{\kappa_y^2}{(2\tilde{\rho} - \rho_m)^2} (\alpha_r z^2 - 1)^2 - \alpha_x (\alpha_r z^2 - 1) - \alpha_x = 0 \\ \Leftrightarrow & \left(\frac{\alpha_r z^2 - 1}{2\tilde{\rho} - \rho_m} \right)^2 - \frac{\alpha_x (2\tilde{\rho} - \rho_m)}{\kappa_y^2} \left(\frac{\alpha_r z^2 - 1}{2\tilde{\rho} - \rho_m} \right) - \frac{\alpha_x}{\kappa_y^2} = 0 \\ \Leftrightarrow & \bar{A}_\pi^2 - \frac{\alpha_x (2\tilde{\rho} - \rho_m)}{\kappa_y^2} \bar{A}_\pi - \frac{\alpha_x}{\kappa_y^2} = 0, \end{aligned} \quad (62)$$

where

$$\bar{A}_\pi \equiv \lim_{\alpha_r \rightarrow 0} A_\pi = \frac{\alpha_r z^2 - 1}{2\tilde{\rho} - \rho_m} \quad (63)$$

by equation (54). Then, by the optimality condition $A_\pi > 0$, we have

$$\bar{A}_\pi = \frac{\alpha_r z^2 - 1}{2\tilde{\rho} - \rho_m} = \frac{\alpha_x(2\tilde{\rho} - \rho_m)}{2\kappa_y^2} + \sqrt{\frac{\alpha_x^2(2\tilde{\rho} - \rho_m)^2}{4\kappa_y^4} + \frac{\alpha_x}{\kappa_y^2}}. \quad (64)$$

The second equality also implies

$$\begin{aligned} \alpha_r z^2 &= 1 + \frac{\alpha_x(2\tilde{\rho} - \rho_m)^2}{2\kappa_y^2} + (2\tilde{\rho} - \rho_m) \sqrt{\frac{\alpha_x^2(2\tilde{\rho} - \rho_m)^2}{4\kappa_y^4} + \frac{\alpha_x}{\kappa_y^2}} \\ &= 1 + \frac{\alpha_x(2\tilde{\rho} - \rho_m)^2}{2\kappa_y^2} + \frac{2\tilde{\rho} - \rho_m}{|2\tilde{\rho} - \rho_m|} \sqrt{\frac{\alpha_x^2(2\tilde{\rho} - \rho_m)^4}{4\kappa_y^4} + \frac{\alpha_x(2\tilde{\rho} - \rho_m)^2}{\kappa_y^2}} \\ &= 1 + \frac{\alpha_x(2\tilde{\rho} - \rho_m)^2}{2\kappa_y^2} + \frac{2\tilde{\rho} - \rho_m}{|2\tilde{\rho} - \rho_m|} \sqrt{\left(1 + \frac{\alpha_x(2\tilde{\rho} - \rho_m)^2}{2\kappa_y^2}\right)^2 - 1} > 0, \end{aligned} \quad (65)$$

thus, the solutions given in (64) and (65) are valid.

Equation (65) shows that $z \sim \alpha_r^{1/2}$ as $\alpha_r \rightarrow 0$. These properties and equations (53) and (55) imply

$$B_{\pi x} \rightarrow \gamma \alpha_r z \sim \alpha_r^{1/2} \quad \text{and} \quad A_x \rightarrow -\gamma \kappa_y \frac{\bar{A}_\pi}{z} \sim \alpha_r^{1/2}$$

as $\alpha_r \rightarrow 0$.

Rewrite equations (57) and (58) as

$$\begin{aligned} -(\rho_m - \tilde{\rho})C_\pi - \frac{1}{\gamma}C_x - \frac{B_{\pi x}}{\gamma} \left(\frac{1}{\gamma \alpha_r} C_x + \tilde{r} \right) &= 0, \\ -\rho_m C_x - \kappa_y C_\pi - \frac{A_x}{\gamma} \left(\frac{1}{\gamma \alpha_r} C_x + \tilde{r} \right) &= 0. \end{aligned}$$

These equations show that $C_\pi \sim \alpha_r$, $C_x \sim \alpha_r$, and

$$\beta_0^* - \tilde{r} = - \left(\frac{1}{\gamma \alpha_r} C_x - \tilde{r} \right) \sim \alpha_r^{1/2}.$$

Otherwise, equations (57) and (58) cannot hold.

These properties show that, as $\alpha_r \rightarrow 0$, A_x , $B_{\pi x}$, C_x , and C_π all converge to zero, and hence

$$\bar{D} \equiv \lim_{\alpha_r \rightarrow 0} D = \frac{1}{2\rho_m} \|\sigma_\pi\|^2 \bar{A}_\pi.$$

Notice that equations (63) and (62) imply

$$\frac{\alpha_r z^2}{\kappa_y \bar{A}_\pi} = \frac{(2\tilde{\rho} - \rho_m)\bar{A}_\pi + 1}{\kappa_y \bar{A}_\pi} = \frac{\kappa_y \bar{A}_\pi}{\alpha_x},$$

thus, as $\alpha_r \rightarrow 0$, we have

$$\begin{aligned} r^* &= -\frac{1}{\gamma\alpha_r} \left(A_x x + B_{\pi x}(\pi - \bar{\pi}) + C_x \right) \\ &\rightarrow \tilde{r} - \frac{1}{\gamma\alpha_r} \left(A_x x + B_{\pi x}(\pi - \bar{\pi}) \right) \\ &\rightarrow \tilde{r} + \frac{1}{\alpha_r z} \left(\kappa_y \bar{A}_\pi x - \alpha_r z^2(\pi - \bar{\pi}) \right) \\ &= \tilde{r} + \frac{\kappa_y \bar{A}_\pi}{\alpha_r z} \left(x - \frac{\alpha_r z^2}{\kappa_y \bar{A}_\pi}(\pi - \bar{\pi}) \right) = \tilde{r} + \frac{\kappa_y \bar{A}_\pi}{\alpha_r z} \left(x - \frac{\kappa_y \bar{A}_\pi}{\alpha_x}(\pi - \bar{\pi}) \right). \end{aligned}$$

Since $\frac{\kappa_y \bar{A}_\pi}{\alpha_r z} \sim \alpha_r^{-1/2}$, the optimal control for the case of $\alpha_r = 0$ is given by (35).

A.2.4 Output Gap Is Directly Controllable

The associated HJB equation with optimal control problem (36) is:

$$-\rho_m V_0 + \min_x \left\{ \left(\tilde{\rho}(\pi - \bar{\pi}) - \kappa_y x \right) \frac{dV_0}{d\pi} + \frac{1}{2} \|\sigma_\pi\|^2 \frac{d^2 V_0}{d\pi^2} + \frac{1}{2} \left((\pi - \bar{\pi})^2 + \alpha_x x^2 \right) \right\} = 0.$$

With a guess of the monetary cost function as

$$V_0(\pi) = \frac{1}{2} A_{\pi 0} (\pi - \bar{\pi})^2 + C_{\pi 0} (\pi - \bar{\pi}) + D_0,$$

and hence the optimal control is

$$x = \frac{\kappa_y}{\alpha_x} \frac{dV_0}{d\pi} = \frac{\kappa_y}{\alpha_x} \left(A_{\pi 0} (\pi - \bar{\pi}) + C_{\pi 0} \right).$$

Then, the optimal control problem is to solve the following three equations:

$$(\pi - \bar{\pi})^2 : -\frac{1}{2} \rho_m A_{\pi 0} + \tilde{\rho} A_{\pi 0} - \frac{1}{2} \frac{\kappa_y^2}{\alpha_x} A_{\pi 0}^2 + \frac{1}{2} = 0,$$

$$\begin{aligned} \pi - \bar{\pi} : \quad & -\rho_m C_{\pi 0} + \tilde{\rho} C_{\pi 0} - \frac{\kappa_y^2}{\alpha_x} A_{\pi 0} C_{\pi 0} = 0, \\ \text{constant} : \quad & -\rho_m D_0 - \frac{1}{2} \frac{\kappa_y^2}{\alpha_x} C_{\pi 0}^2 + \frac{1}{2} \|\sigma_\pi\|^2 A_{\pi 0} = 0. \end{aligned}$$

Notice that $C_{\pi 0} = 0$, $A_{\pi 0} = \bar{A}_\pi$ and $D_0 = \bar{D}$. Then the cost function is exactly the same as taking $\alpha_r = 0$ in the general case discussed in Appendix A.2.3.

A.3 Proofs

Proof of Proposition 1 Negativity of β_x^* is trivial. It is direct implication of optimality of monetary policy.

The optimality implies that both A_π and A_x are positive. The HJB equation for term $(\pi - \bar{\pi})^2$, equation (50), is equivalent to

$$z \equiv \frac{1}{\gamma \alpha_r} B_{\pi x} + 1 = \pm \sqrt{1 + \frac{1 + (2\tilde{\rho} - \rho_m) A_\pi}{\alpha_r}}.$$

This shows that z and $B_{\pi x}$ have the same sign if $\rho_m \leq 2\tilde{\rho}$ because A_π is positive. On the other hand, the HJB equation for term $(\pi - \bar{\pi})x$, equation (51), can be rewritten as

$$(\rho_m - \tilde{\rho}) B_{\pi x} + \kappa_y A_\pi + \frac{1}{\gamma} A_x \left(\frac{1}{\gamma \alpha_r} B_{\pi x} + 1 \right) = 0.$$

This shows, given $\rho_m \geq \tilde{\rho}$, that z and hence $B_{\pi x}$ cannot be positive because A_π and A_x are positive, thus

$$\frac{1}{\gamma \alpha_r} B_{\pi x} + 1 = -\sqrt{1 + \frac{1 + (2\tilde{\rho} - \rho_m) A_\pi}{\alpha_r}}. \quad (66)$$

This also implies

$$\beta_\pi^* = -\frac{1}{\gamma \alpha_r} B_{\pi x} = 1 + \sqrt{1 + \frac{1 + (2\tilde{\rho} - \rho_m) A_\pi}{\alpha_r}} > 1.$$

This inequality with the interest rule (28) is known as *Taylor Principle* in the literature.

Proof of Proposition 2 The dynamics of inflation and output gaps given by equation (30) are stable when the two eigenvalues of the κ -matrix are negative. The eigenvalues of the κ -matrix are

the solutions to

$$\det \left| \begin{bmatrix} \tilde{\rho} & -\kappa_y \\ \frac{1}{\gamma}(\beta_\pi - 1) & \frac{1}{\gamma}\beta_x \end{bmatrix} - bI \right| = 0,$$

where I is the identity matrix. After some algebraic manipulations, we have

$$b = \frac{1}{2} \left\{ \left(\tilde{\rho} + \frac{\beta_x}{\gamma} \right) \pm \sqrt{\left(\tilde{\rho} + \frac{\beta_x}{\gamma} \right)^2 + \frac{4}{\gamma} \left(-\tilde{\rho}\beta_x + \kappa_y(-\beta_\pi + 1) \right)} \right\}.$$

Thus, the necessary and sufficient conditions for both of the eigenvalues being negative (or real parts) are

$$\tilde{\rho} + \frac{\beta_x}{\gamma} < 0, \quad -\tilde{\rho}\beta_x + \kappa_y(-\beta_\pi + 1) < 0.$$

These conditions are equivalent to the ones stated in the proposition.

Proof of Proposition 3 The zero coefficients directly follow from the solutions to the general case presented in section A.2.1. First note that equation (56) shows $z \rightarrow \frac{\tilde{\rho} - \rho_m}{2\tilde{\rho} - \rho_m} \rightarrow 1$ when $\rho_m \rightarrow \infty$. Then, both A_x and $B_{\pi x}$ also converge to 0 by equations (55) and (53). These then imply C_x goes to 0 by equation (59).

Proof of Propositions 4 and 5 The proofs Propositions 4 and 5 are in Appendix A.2.3 and A.2.4, respectively.

Proof of Proposition 6 For ease of notation and simplicity, we only show the results for $K = 2$. The proof can be straightforwardly extended for any $K > 2$.

Given an arbitrary monetary policy rule $\{r_s\}$, then the associated loss function is

$$J(t) = E_t \left[\int_t^\infty e^{-\int_t^s \rho_m(a-) da} L_s ds \right], \quad (67)$$

where

$$L_s = \frac{1}{2} \left((\pi_s - \bar{\pi})^2 + \alpha_x x_s^2 + \alpha_r r_s^2 \right).$$

Let $J_k(t)$ be the loss function under regime k , in which $\rho_m(t-) = \rho_{mk}$. By the Martingale Repr-

sensation Theorem (e.g., see [Karatzas and Shreve \(1991\)](#)), equation (67) implies that

$$e^{-\int_0^t \rho_m(s^-) ds} J(t) + \int_0^t e^{-\int_0^s \rho_m(a^-) da} L_s ds$$

is a martingale. An application of Itô's Lemma to the above shows

$$\begin{aligned} -\rho_{m1} J_1 + \text{drift of } J_1 + Q_{12}(J_2 - J_1) + L &= 0, \\ -\rho_{m2} J_2 + \text{drift of } J_2 + Q_{21}(J_1 - J_2) + L &= 0. \end{aligned}$$

Since $Q_{11} = -Q_{12}$ and $Q_{22} = -Q_{21}$, we can rewriting the system of equations in vector (bold) and matrix form as

$$A\mathbf{J} + \text{drift of } \mathbf{J} + L\mathbf{1} = \mathbf{0}, \quad (68)$$

where A (as defined by (37)) is a constant matrix and can be decomposed into $A = \Phi\Lambda\Phi^{-1}$, where Φ is a constant matrix formed by eigenvectors of A , and Λ is a diagonal matrix and its diagonal elements are the eigenvalues of A . Then, equation (68) and the transversality condition $\lim_{t \rightarrow \infty} e^{At} \mathbf{J}(t) \rightarrow \mathbf{0}$ imply

$$\begin{aligned} \mathbf{J}(t) &= E_t \left[\int_t^\infty e^{A(s-t)} L_s \mathbf{1} ds \right] \\ &= E_t \left[\int_t^\infty \Phi^{-1} \exp(\Lambda(s-t)) \Phi L_s \mathbf{1} ds \right] \\ &= \Phi^{-1} \begin{bmatrix} E_t \left[\int_t^\infty e^{\Lambda_{11}(s-t)} L_s ds \right] & 0 \\ 0 & E_t \left[\int_t^\infty e^{\Lambda_{22}(s-t)} L_s ds \right] \end{bmatrix} \Phi \mathbf{1}. \end{aligned}$$

Thus, the original optimization problem is equivalent to the following:

$$\min_{\{r_s\}} \mathbf{J}(t) = \Phi^{-1} \begin{bmatrix} \min_{\{r_s\}} E_t \left[\int_t^\infty e^{\Lambda_{11}(s-t)} L_s ds \right] & 0 \\ 0 & \min_{\{r_s\}} E_t \left[\int_t^\infty e^{\Lambda_{22}(s-t)} L_s ds \right] \end{bmatrix} \Phi \mathbf{1}.$$

These equations show that the optimal monetary policy under regime k is equivalent to the optimal single-regime policy with the monetary discount rate $-\Lambda_{kk}$. It is straightforward to extend these arguments into the case with any $K > 2$.

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